0.1: The Binomial Theorem and Pascal's Triangle.

The formulas

$$(x+y)^2 = x^2 + 2xy + y^2, \text{ and}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

are special instances of the **Binomial Theorem**, which says that the coefficients $\square$ in the expansion

$$(x+y)^n = \square x^n + \square x^{n-1}y + \square x^{n-2}y^2 + \cdots + \square x^2y^{n-2} + \square xy^{n-1} + \square y^n$$

are found in **Pascal's Triangle**:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pascal's Triangle has infinitely many rows. We refer to the top row as its 0th row. Except for the 1's that begin and end each row, each entry in the triangle is the sum of the two entries directly above.
See http://kunklet.people.cofc.edu/MATH111/pascal.pdf for more details.

0.1.re1. The third row "1 3 3 1" and the fourth row "1 4 6 4 1" tell us that

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

0.1.re2. Generate the fifth, sixth, and seventh rows of Pascal's Triangle.

When we expand $(x-y)^n$, the successive terms alternate in sign:

0.1.re3.

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

0.1.re4. Expand the following.

a. $(x+3)^4$ b. $(u+v)^5$ c. $(u-v)^6$
d. $(x^3+y)^4$ e. $(x-x^{-1})^5$ f. $(\xi-2)^6$

Answers

0.1.re2. 5th row: 1 5 10 10 5 1. 6th row: 1 6 15 20 15 6 1. 7th row: 1 7 21 35 35 21 7 1.

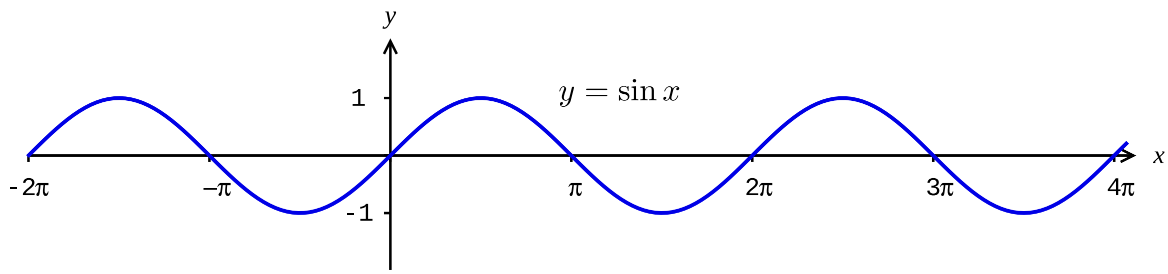
0.1.re4a. $x^4 + 12x^3 + 54x^2 + 108x + 81$ 0.1.re4b. $u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5$

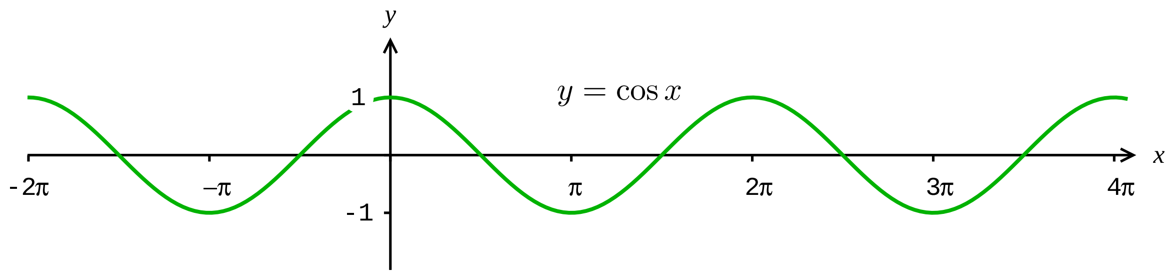
0.1.re4c. $u^6 - 6u^5v + 15u^4v^2 - 20u^3v^3 + 15u^2v^4 - 6uv^5 + v^6$ 0.1.re4d. $x^{12} + 4x^9y + 6x^6y^2 + 4x^3y^3 + y^4$

0.1.re4e. $x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} + x^{-5}$ 0.1.re4f. $\xi^6 - 12\xi^5 + 60\xi^4 - 160\xi^3 + 240\xi^2 - 192\xi + 64$

Ap.D: Trigonometry

For a more complete review of trignometry, see Appendix D of our text.
The two basic functions in trigonometry are the sine and cosine, graphed here:



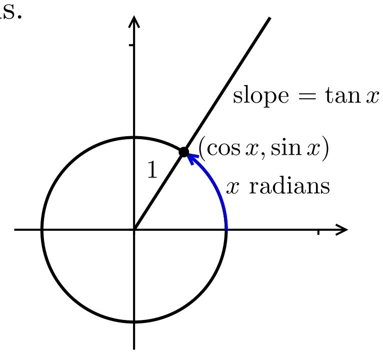


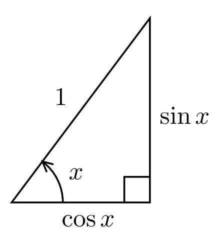
The other four trig functions are defined using sine and cosine:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

$\sin x$ and $\cos x$ are defined for all real numbers x, but $\tan x$ and $\sec x$ are undefined whenever $\cos x = 0$, and $\cot x$ and $\csc x$ are undefined whenever $\sin x = 0$.
By definition, $\cos x$ and $\sin x$ are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin) x radians counterclockwise from the positive horizontal axis.



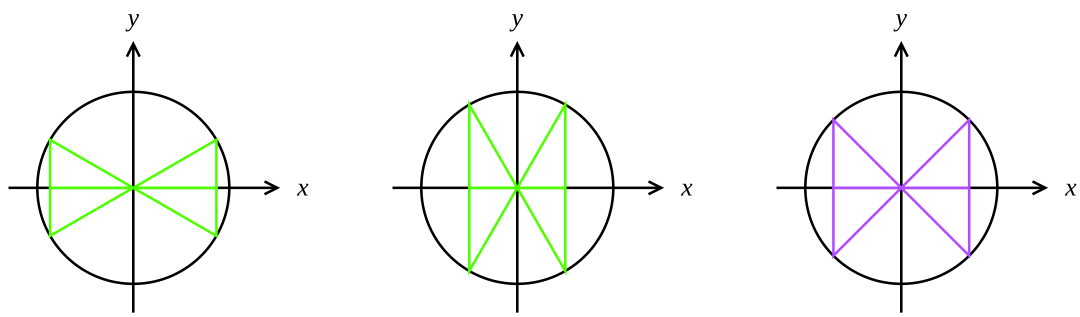


Consequently, the ray through the origin x radians from the positive horizontal axis has slope $\tan x$, and, when x is an acute angle, $\cos x$ and $\sin x$ are the legs of this right triangle with hypotenuse 1 and interior angle x

Known values of sine and cosine

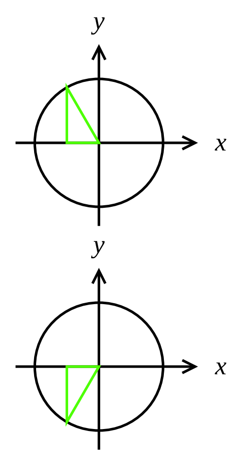
We already know the values of sine and cosine at the four points where the unit circle intersects the x and y axes. By placing these two triangles:

around the unit circle like this:



we find the sines and cosines at 12 more points on the unit circle (and the infinitely many angles that reach those points).

Ap.D.re1. Find all angles whose cosine is $-\frac{1}{2}$.



We recognize $1/2$ as the short side of the 30-60-90 triangle, so for the cosine to be $-1/2$, the angle must be one of the two pictured at right. Find one angle that matches each drawing, for instance,

$$x = \pi - \pi/3 = 2\pi/3 \quad \text{and} \quad x = \pi + \pi/3 = 4\pi/3.$$

Then add all multiples of 2π to describe *all* angles that fit the drawings:

$$x = 2\pi/3 + 2\pi n \quad \text{and} \quad x = 4\pi/3 + 2\pi n \quad \text{(where } n \text{ is any integer).}$$

Ap.D.re2. Find all solutions x to the given equation.

a. $\sin x = -1/\sqrt{2}$ b. $\cos x = 0$ c. $\tan x = -1$

d. $\sec x = -2$ e. $\csc x = 2/\sqrt{3}$ f. $\cot x = \sqrt{3}$

Identities

You must know the derivatives of the six trig functions and these trig identities:

PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$
EVEN & ODD IDENTITIES
$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$
SUM & DIFFERENCE IDENTITIES
$\sin(x+y) = \sin x \cos y + \cos x \sin y$ $\cos(x+y) = \cos x \cos y - \sin x \sin y$ $\sin(x-y) = \sin x \cos y - \cos x \sin y$ $\cos(x-y) = \cos x \cos y + \sin x \sin y$
DOUBLE ANGLE FORMULAS
$\sin(2x) = 2 \sin x \cos x$ $\cos(2x) = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$
HALF ANGLE FORMULAS
$\cos^2 x = \frac{1}{2}\big(1 + \cos(2x)\big)$ $\sin^2 x = \frac{1}{2}\big(1 - \cos(2x)\big)$

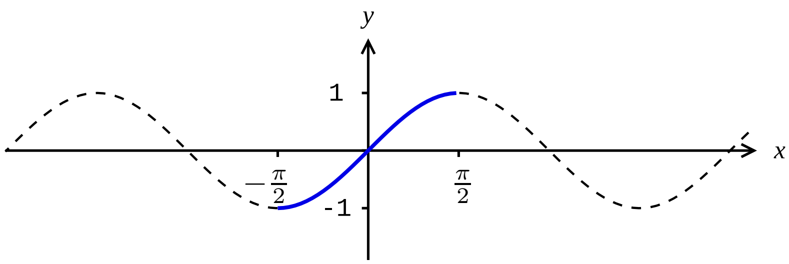
Ap.D.re3. The first pythagorean identity follows from the definition of sine and cosine. Derive the second and third pythagorean identities from the first by dividing by either $\sin^2 x$ or $\cos^2 x$.

Ap.D.re4. The formula for $\cos(x+y)$ is obtained by differentiating both sides of the formula for $\sin(x+y)$ with respect to x assuming y is a constant. Derive the other sum/difference/double/half-angle formulas from these two by differentiating with respect to x, replacing y with $-y$ and using the even/odd identities, or replacing y with x.

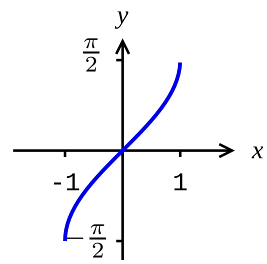
The inverse trig functions

To define inverses of the trig functions, we restrict each to a domain on which it takes each value in its range exactly once. The three we'll see most often in calculus are the inverses of the sine, tangent, and secant.

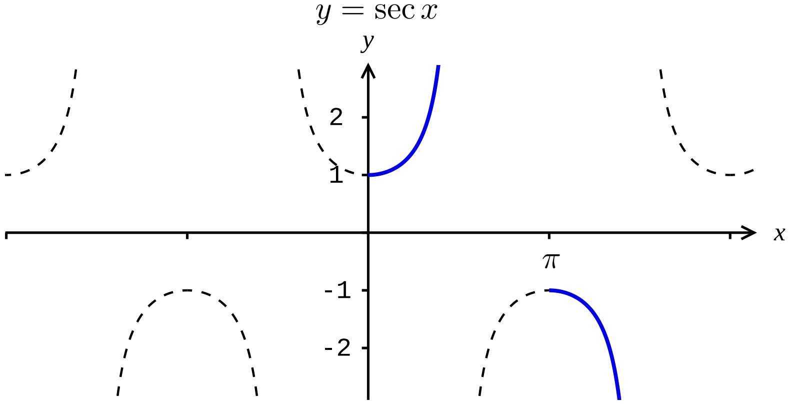
$y = \sin x$



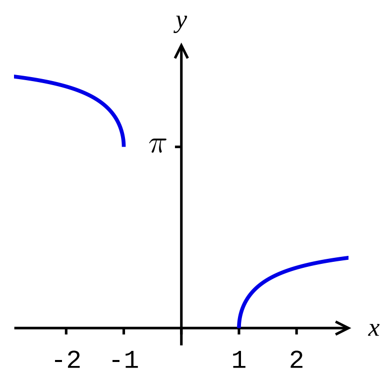
$y = \sin^{-1} x$



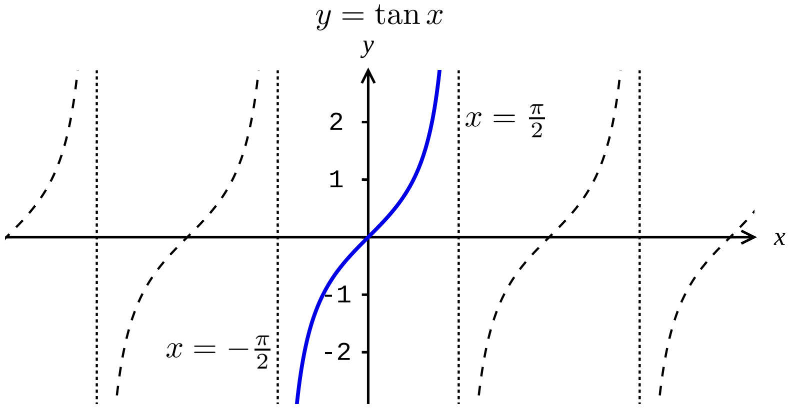
$y = \sec x$



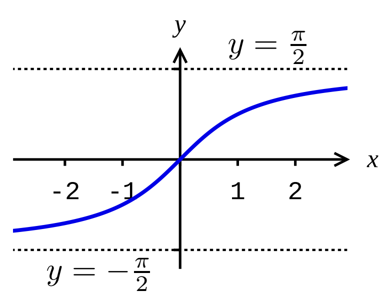
$y = \sec^{-1} x$



$y = \tan x$



$y = \tan^{-1} x$



Traditionally, each of the inverse trig functions has two names. The inverse function of $\sin x$ is called $\sin^{-1} x$ or $\arcsin x$, the inverse function of $\sec x$ is called $\sec^{-1} x$ or $\operatorname{arcsec} x$, etc..

As indicated in the graphs above,

	Domain	Range
$\sin^{-1}$	$[-1,1]$	$\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\dfrac{\pi}{2}\right)\cup\left[\pi,\dfrac{3\pi}{2}\right)$
$\tan^{-1}$	$(-\infty,\infty)$	$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

although there are few times in this course when you'll need to know these details.
Tip: if $x>0$, and "arctrig" is any of the inverse trig functions, then $\operatorname{arctrig} x \in [0,\frac{\pi}{2}]$.

It is helpful to remember the definitions of the inverse trig functions in words:

If $-1\le x\le 1$, then $\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is x. That is,

$$\sin(\sin^{-1} x)=x \quad \text{for any } x\in[-1,1].$$

If $|x|\ge 1$, then $\sec^{-1} x$ is the angle in $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ whose secant is x. That is,

$$\sec(\sec^{-1} x)=x \quad \text{for any } x\in(-\infty,-1]\cup[1,\infty).$$

If $-\infty<x<\infty$, then $\tan^{-1} x$ is the angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x. That is,

$$\tan(\tan^{-1} x)=x \quad \text{for any real number } x.$$

Ap.D.re5. Evaluate: $\sin^{-1}\left(-\frac{1}{2}\right)$

Answers

Ap.D.re2a. $x=-\pi/4+2\pi n$ or $x=-3\pi/4+2\pi n$ Ap.D.re2b. $x=\pi/2+2\pi n$ or $x=-\pi/2+2\pi n$
Ap.D.re2c. $x=3\pi/4+2\pi n$ or $x=11\pi/4+2\pi n$ Ap.D.re2d. (same as in example Ap.D.re1)
Ap.D.re2e. $x=\pi/3+2\pi n$ or $x=2\pi/3+2\pi n$ Ap.D.re2f. $x=\pi/3+2\pi n$ or $x=4\pi/3+2\pi n$
Ap.D.re3. Hint: $\frac{\sin^2}{\cos^2 x}=\tan^2 x$. $\frac{1}{\cos^2 x}=\sec^2 x$. $\frac{\cos^2}{\sin^2 x}=\cot^2 x$. $\frac{1}{\sin^2 x}=\csc^2 x$.
Ap.D.re4. Hint: Obtain $\sin(x-y)$ [or $\cos(x-y)$] from $\sin(x+y)$ [or $\cos(x+y)$] by replacing y with $-y$ and using $\sin(-y)=-\sin(y)$ and $\cos(-y)=\cos y$. Obtain $\sin(2x)$ [or the first form of $\cos(2x)$] from $\sin(x+y)$ [or $\cos(x+y)$] by replacing y with x. Obtain other forms of $\cos(2x)$ using $\cos^2 x+\sin^2 x=1$. Obtain the half angle formula for $\cos^2 x$ [or $\sin^2 x$] from the second [or third] double angle formula for $\cos(2x)$.
Ap.D.re5. $-\pi/6$

D: Differentiation

The **derivative** of the function $f(x)$, denoted $f'(x)$, is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

The **linearization** of a function $f(x)$ at $x = a$ is the linear function having the same value and derivative value as f at a and is given by

$$L(x) = f(a) + f'(a)(x - a).$$

Differentiation rules allow us to differentiate many functions without resorting to the definition. These rules come in two types.

I. A catalog of elementary functions and their derivatives.

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\|x\|$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

II. Combination laws.

Linearity Rules: If $f(x)$ and $g(x)$ are differentiable and c is a constant, then

$$\big(cf(x)\big)' = cf'(x)$$
$$\big(f(x)+g(x)\big)' = f'(x)+g'(x)$$

Product Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their product, and

$$\big(f(x)g(x)\big)' = f'(x)g(x)+f(x)g'(x).$$

Quotient Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their quotient, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2} \qquad \text{(provided } g(x)\neq 0\text{).}$$

Chain Rule: If f and g are differentiable functions, then their composition is differentiable (where it exists), and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big)g'(x)$$

When more than one combination law applies, remember this advice:

The last operation determines the first differentiation rule we must use.

D.re1. Find the indicated derivatives of the given function.

a. $f'(x)$ & $f''(x)$, where $f(x) = 7x^3 - \frac{3}{\sqrt[3]{x^2}} + \frac{2x}{3x+4}$

b. $g^{(3)}(x)$ & $g^{(4)}(x)$, where $g(x) = 7x^3 + 4x^2 + 2x + \frac{2}{4-x}$

D.re2. Find the quadratic function $k(x)$ satisfying

$$k(2) = 1 \qquad k'(2) = -1 \qquad k''(2) = 6.$$

D.re3. Find the derivative of the given function.

a. $x^2 e^x$ b. $x^2 e^x(2x-3)$ c. $\dfrac{3x^3+2}{x^2}$

d. $\dfrac{x}{3x^2+1}$ e. $e^x\sqrt{x}\sin x$ f. $\dfrac{x^2}{3x^3+2}$

g. $\dfrac{e^x \sin x + 3\cos x}{1-\tan x}$

D.re4. Find the equation of the line tangent to $y = \dfrac{xe^x}{3x^2-1}$ at $x = 1$.

D.re5. Find the derivative of the given function.

a. $\dfrac{2x-1}{2x+1}$ b. $\dfrac{2x+1}{2x^2-1}$ c. $\dfrac{2x+1}{(2x-1)^2}$

d. $\sin(x^2)$ e. $\sin^2 x$ f. $\sqrt{x^2-5x+1}$

g. $\ln 2$ h. $\ln(2x)$ i. $\ln(e^{-x}5x^2\sin x)$

j. $\ln|\sec x|$ k. $\log_3 x$ l. $e^{\frac{1}{2}\ln x}$

m. $x^{x/2}$ n. $e^{x\sin x}$ o. 2^x

p. $e^{\tan^2 x}$ q. $x^2\cos^2 x^3$ r. $(\sin^2 x)e^{\cos x}$

s. $\frac{e^x}{\sec\sqrt{x}}$ t. $\tan^{-1}(\sin^{-1} x)$ u. $(\tan^{-1} x)(\sin^{-1} x)$

v. $e^x \arccos x^3$ w. $\sec(\sec^{-1} x)$ x. $\cot(\arctan e^x)$

D.re6. Find the linearization of the function at the given point.

a. $e^x,\ a = 0$ b. $e^x,\ a = 1$

c. $\sin x,\ a = 0$ d. $\sin x,\ a = \pi/4$

e. $\sqrt{x},\ a = 9$ f. $e^x(\sin x - \cos x)$

For more differentiation practice, see the review problems from Chapter 3 listed on our syllabus under **Assigned Problems**.

Answers

D.re1a. $f'(x) = 21x^2 + 2x^{-5/3} + \frac{8}{3(3x+4)^2}$. $f''(x) = 42x - \frac{10}{3}x^{-8/3} - \frac{16}{(3x+4)^3}$. D.re1b. $g'''(x) = 42 + 12(4-x)^{-4}$. $g^{(4)}(x) = 48(4-x)^{-5}$. D.re2. $3x^2 - 13x + 15$ D.re3a. $(x^2 + 2x)e^x$ D.re3b. $(2x^3 + 3x^2 - 3x)e^x$ D.re3c. $3 - 4x^{-3}$ D.re3d. $\frac{1-3x^2}{(3x^2+1)^2}$ D.re3e. $e^x\sqrt{x}\sin x + \frac{1}{2}e^x\frac{1}{\sqrt{x}}\sin x + e^x\sqrt{x}\cos x$ D.re3f. $\frac{-3x^4+4x}{(3x^3+2)^2}$ D.re3g. $\frac{\left(e^x(\sin x + \cos x) - 3\sin x\right)(1-\tan x) + (e^x\sin x + 3\cos x)(\sec^2 x)}{(1-\tan x)^2}$ D.re4. $y - \frac{e}{2} = -\frac{e}{2}(x-1)$ D.re5a. $\frac{4}{(2x+1)^2}$ D.re5b. $\frac{-4x^2-4x-2}{(2x^2-1)^2}$ D.re5c. $\frac{-4x-6}{(2x-1)^3}$ D.re5d. $2x\cos(x^2)$ D.re5e. $2\sin x\cos x$ D.re5f. $\frac{1}{2}(2x-5)(x^2-5x+1)^{-1/2}$ D.re5g. 0 D.re5h. x^{-1} D.re5i. $-1 + \frac{2}{x} + \cot x$ D.re5j. $\tan x$ D.re5k. hint: $\log_3 x = (\ln x)/(\ln 3)$. Derivative $= \frac{1}{x\ln 3}$. D.re5l. hint: simplify before differentiating. Derivative is $\frac{1}{2}x^{-1/2}$. D.re5m. hint: $a^b = e^{\ln a^b} = e^{b\ln a}$. Derivative $= \frac{1}{2}e^{\frac{1}{2}x\ln x}(1 + \ln x)$. D.re5n. $(x\cos x + \sin x)e^{x\sin x}$ D.re5o. hint: $2^x = e^{\ln(2^x)} = e^{x\ln 2}$. Derivative $= e^{x\ln 2}\ln 2$. D.re5p. $e^{\tan^2 x}2\tan x\sec^2 x$ D.re5q. $2x\cos^3(x^3) - 6x^4\cos(x^3)\sin(x^3)$ D.re5r. $(2\sin x\cos x - \sin^3 x)e^{\cos x}$ D.re5s. hint: $\frac{e^x}{\sec\sqrt{x}} = e^x\cos\sqrt{x}$. Derivative $= e^x(\cos(\sqrt{x}) - \frac{1}{2}x^{-1/2}\sin(\sqrt{x}))$. D.re5t. $\frac{1}{(1+(\sin^{-1}x)^2)\sqrt{1-x^2}}$ D.re5u. $\frac{\sin^{-1}x}{1+x^2} + \frac{\tan^{-1}x}{\sqrt{1-x^2}}$ D.re5v. $e^x\left(\cos^{-1}x^3 - \frac{3x^2}{\sqrt{1-x^6}}\right)$ D.re5w. hint: simplify before differentiating. Derivative $= 1$. D.re5x. hint: simplify before differentiating. Derivative is $-e^{-x}$. D.re6a. $1 + x$ D.re6b. xe D.re6c. x D.re6d. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4})$ D.re6e. $3 + \frac{1}{6}(x-9)$ D.re6f. $e^{\pi/2}\Big(1 + 2(x - \pi/2)\Big)$

I: Integration

The **definite integral** from a to b of $f(x)\,dx$, written $\int_a^b f(x)\,dx$, is the limit of the Riemann sums of f on the interval $[a,b]$ as the length of the subintervals goes to zero.

The **indefinite integral** of $f(x)\,dx$, written $\int f(x)\,dx$, is the collection of all **antiderivatives** of $f(x)$, that is, those functions $F(x)$ whose derivative $F'(x)$ equals $f(x)$.

$$\begin{aligned}
\int_a^b f(x)\,dx &= \text{a number.}\\
&= \text{the net signed area trapped between the graph of } f(x)\\
&\quad\ \text{and the } x\text{-axis from } x=a \text{ to } x=b.\\
&= \text{the net distance traveled from time } x=a \text{ to time } x=b\\
&\quad\ \text{by an object whose velocity at time } x \text{ is } f(x).\\
\int f(x)\,dx &= \text{a collection of functions.}\\
&= \text{the set of all antiderivatives of } f(x).
\end{aligned}$$

If $F(x)$ is an antiderivative of $f(x)$ on an interval, then, by the Mean Value Theorem, every antiderivative of f on that interval equals $F(x)+C$ for some constant C:

$$\int f(x)\,dx = F(x)+C, \text{ where } F'(x)=f(x)$$

According to the FTC below, evaluating a definite integral consists primarily of finding an antiderivative. We refer to both of those operations as **integration**.

Fundamental Theorem of Calculus: If $F(x)$ is any antiderivative of $f(x)$ and if f is continuous on $[a,b]$, then

$$\int_a^b f(t)\,dt = F(b)-F(a) \stackrel{\text{def}}{=} F(x)\Big|_a^b$$

I.re1. Integrate.

a. $\displaystyle\int (x^{0.7}-2e^x)\,dx$ b. $\displaystyle\int_1^2 (1-6t^2+10t^4)\,dt$ c. $\displaystyle\int \frac{r^3+1}{2r}\,dr$

d. $\displaystyle\int_{-\pi/6}^{\pi/4} \sec x\tan x\,dx$ e. $\displaystyle\int_{-3}^{3} x^{11}\,dx$ f. $\displaystyle\int (u+3)(u-4)\,du$

A table of antiderivatives is practically the same as a table of derivatives read backwards:

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\lvert x\rvert$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

$f(x)$	antiderivative of f
$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1}+C & \text{if } n \neq -1 \\ \ln\lvert x\rvert + C & \text{if } n = -1 \end{cases}$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$

There are fewer combination rules for antiderivatives than for derivatives. If f and g are functions and c is a constant, then

$$\boxed{\begin{aligned} \int \big(f(x)+g(x)\big)\,dx &= \int f(x)\,dx + \int g(x)\,dx \\ \int cf(x)\,dx &= c\int f(x)\,dx \end{aligned}}$$

Generally, it's difficult to integrate a product or quotient, because

$$\left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right) \neq \int f(x)g(x)\,dx, \quad \text{and}$$
$$\left(\int f(x)\,dx\right) \div \left(\int g(x)\,dx\right) \neq \int \big(f(x)/g(x)\big)\,dx.$$

For example, x^2 is an antiderivative for $2x$, and e^x is an antiderivative for itself, but x^2e^x is **not** an antiderivative for $2xe^x$, and $\dfrac{e^x}{x^2}$ is **not** an antiderivative for $\dfrac{e^x}{2x}$.

I.re2. Find $f(x)$ if

a. $f'(x) = \dfrac{1}{x}$ and $f(-2) = 1$

b. $f''(x) = \sin x - x^2$ and $f'(0) = 2$ and $f(0) = -1$

I.re3. Integrate.

a. $\displaystyle\int (x^2 - 3)\sqrt{x}\,dx$ b. $\displaystyle\int_2^1 \frac{x^2-3}{x^{-2}}\,dx$ c. $\displaystyle\int_0^1 e^p\,dp$

d. $\displaystyle\int_{-6}^{-2} \frac{ds}{s}$ e. $\displaystyle\int_{-1}^{4} e\,dw$ f. $\displaystyle\int (2\csc x \cot x - 3\cos x)\,dx$

g. $\displaystyle\int (x^2-1)^3\,dx$ h. $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 t}\,dt$ i. $\displaystyle\int \frac{r^3 - 7r^2 + 1}{2r}\,dr$

j. $\displaystyle\int \frac{x^3+8}{x+2}\,dx$ k. $\displaystyle\int_{-2}^{2} |x|\,dx$ l. $\displaystyle\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$

m. $\displaystyle\int \left(\frac{2}{x} + \frac{3}{1+x^2}\right)dx$ n. $\displaystyle\int \csc x(\csc x - \cot x)\,dx$

The **Substitution Rule** uses the chain rule to help us to simplify an integral. It says that an integral of the form $\int f\big(u(x)\big)u'(x)\,dx$ (for some function u of x) can be rewritten

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

and thenceforth be treated as if u were the variable of integration. For this reason, substitution is also called a **change of variable**.

I.re4. Integrate.

a. $\displaystyle\int x^3 \sin(x^4)\,dx$ b. $\displaystyle\int (2x+1)\sqrt{x^2+x}\,dx$ c. $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$

d. $\displaystyle\int 2^{x+3}\,dx$ e. $\displaystyle\int \cos x \cot(\sin x)\sec^2(\sin x)\,dx$ f. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

g. $\displaystyle\frac{2e^{\sqrt{x}}}{\sqrt{x}}\,dx$ h. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$ i. $\displaystyle\int (2t+1)^{19}\,dt$

j. $\displaystyle\int x\sqrt{x+2}\,dx$

For more integration practice, see the review problems from Chapter 5 listed on our syllabus under **Assigned Problems**.

Answers

I.re1a. $\frac{1}{1.7}x^{1.7} - 2e^x + C$ I.re1b. 49 I.re1c. $\frac{1}{6}r^3 + \frac{1}{2}\ln|r| + C$ I.re1d. $2+\sqrt{2}$. I.re1e. 0 I.re1f. $\frac{1}{3}u^3 - \frac{1}{2}u^2 - 12u + C$ I.re2a. $f(x) = \ln|x| + 1 - \ln 2$ I.re2b. $f(x) = -\sin x - \frac{1}{12}x^4 + 3x - 1$ I.re3a. $\frac{2}{7}x^{3.5} - 2x^{1.5} + C$ I.re3b. 4/5 I.re3c. $e-1$ I.re3d. $-\ln 3$ I.re3e. $5e$ I.re3f. $-2\csc x - 3\sin x + C$ I.re3g. $\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C$

I.re3h. 1 I.re3i. $\frac{1}{6}r^3\frac{7}{4}r^2-\frac{1}{2}\ln|r|+C$ I.re3j. hint: use long division to rewrite integrand. $\frac{1}{3}x^3-x^2+4x+C$. I.re3k. 4 I.re3l. $\pi/6$ I.re3m. $2\ln|x|+3\tan^{-1}x+C$ I.re3n. $-\cot x+\csc x+C$ I.re4a. $-\frac{1}{4}\cos(x^4)+C$ I.re4b. $\frac{2}{3}(x^2+x)^{3/2}+C$ I.re4c. $\frac{2}{3}(\ln x)^{3/2}+C$ I.re4d. $\frac{1}{\ln 2}2^{x+3}+C$ I.re4e. $\ln|\tan(\sin x)|+C$ I.re4f. $2\sqrt{x}-2\ln(1+\sqrt{x})+C$ I.re4g. $4e^{\sqrt{x}}+C$ I.re4h. $\tan^{-1}e^x+C$. I.re4i. $\frac{1}{40}(2t+1)^{20}+C$ I.re4j. $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+C$

3.11: The Hyperbolic Trig Functions

Definition 3.11.re.1. *The hyperbolic sine and cosine of x are*

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

The other four hyperbolic trig functions are defined in terms of these:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Similarities of the hyperbolic trig functions to the circular trig functions

1. $\cosh 0 = 1$ $\qquad \sinh 0 = 0$
2. $\cosh x$ is an even function, meaning $\cosh(-x) = \cosh x$.

   $\sinh x$ is an odd function, meaning $\sinh(-x) = -\sinh x$.
3. $\cosh^2 x - \sinh^2 x = 1$

   $\cosh x \cosh y + \sinh x \sinh y = \cosh(x + y)$

   $\sinh x \cosh y + \cosh x \sinh y = \sinh(x + y)$

3.11.re1. Use the definitions of sinh and cosh to verify the three identities above.

4.

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

3.11.re2. Find the derivative of the given function.

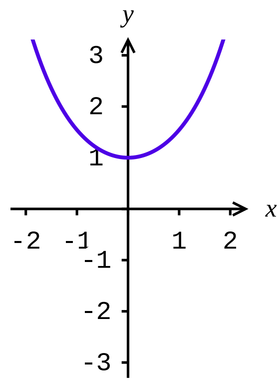
a. $2\sqrt{x} \tanh \sqrt{x}$

b. $\ln(\sinh x)$

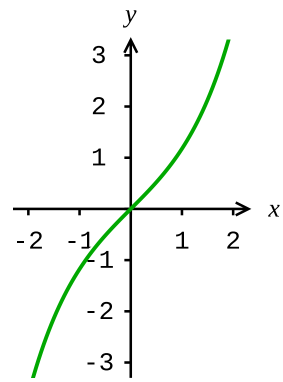
c. $\cos(\sinh x) \sin(\cosh x)$

d. $\ln(\cosh x) - \frac{1}{2} \tanh^2 x$

Graphs of the hyperbolic trig functions



$y = \cosh x$



$y = \sinh x$

Some limits involving the hyperbolic trig functions

3.11.re3. Evaluate the limits, if they exist. It may hep to rewrite the function in terms of exponentials.

a. $\lim_{x\to\infty} \cosh x$ b. $\lim_{x\to-\infty} \cosh x$ c. $\lim_{x\to\infty} \sinh x$

d. $\lim_{x\to-\infty} \sinh x$ e. $\lim_{x\to\infty} \tanh x$ f. $\lim_{x\to-\infty} \tanh x$

Answers

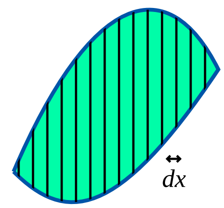
3.11.re2a. $\frac{1}{\sqrt{x}} \tanh \sqrt{x} + \operatorname{sech}^2 \sqrt{x}$ 3.11.re2b. $\coth x$

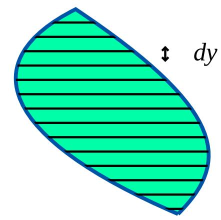
3.11.re2c. $-\cosh x \sin(\sinh x) \sin(\cosh x) + \sinh x \cos(\sinh x) \cos(\cosh x)$ 3.11.re2d. $\tanh^3 x$

3.11.re3a. ∞ 3.11.re3b. ∞ 3.11.re3c. ∞ 3.11.re3d. $-\infty$ 3.11.re3e. 1 3.11.re3f. -1

6.1: Areas Between Curves

To find the area of a region between curves, slice the region with vertical [horizontal] cuts into infinitely many rectangles, each of which is infinitesimally small. Let dA be the area of the generic rectangle at position x [y] having base dx [dy]. The total area is the definite integral of dA:





$$A = \int dA$$

6.1.re1. Find the area of the region bounded by the curves.

a. $y = x(x+4)$ and $y = 6 - x^2$

b. $y = \ln x$, $x = e$, and $y = 0$.

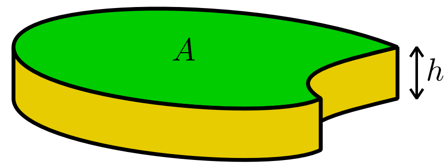
Answers

6.1.re1a. $\frac{64}{3}$ 6.1.re1b. 1

6.2: Volumes

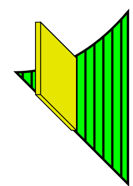
Calculating volumes by slicing

Tip: The volume of a *cylinder* (and not just a circular one) is the area of its base times its height (measured perpendicular to the base).



Volume $= A \cdot h$

6.2.re1. Let R be the region in the first quadrant bounded by the curves $x + y = 1$, $x = 1$, and $y = \cosh x$. Find the volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares. Write your answer as a definite integral, but do not evaluate.



For more of these slicing problems, see
http://kunklet.people.cofc.edu/MATH220/stew0602prob.pdf

Volumes of Revolution: the method of discs and washers

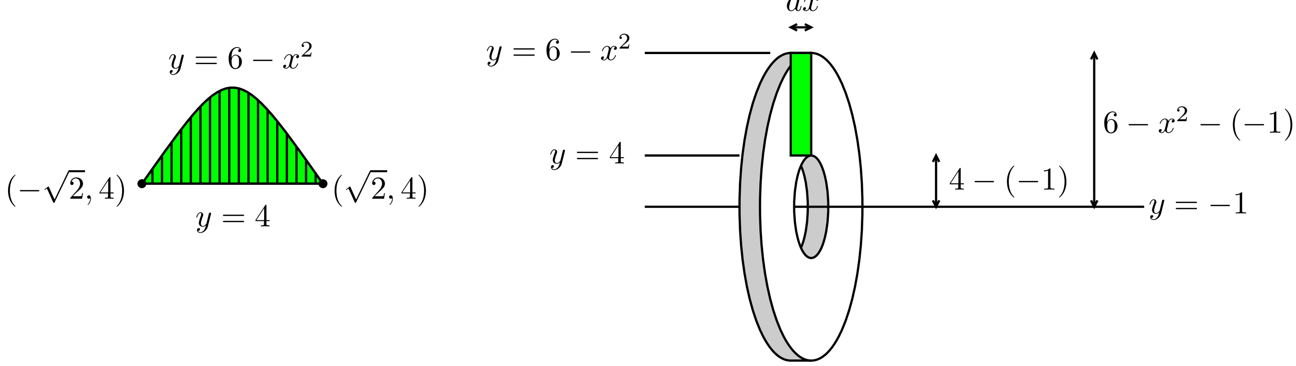
Slicing a region into rectangles and rotating these about a line perpendicular to the slices results in **washers**, or **discs**, if there's no hole. Either is a cylinder, and its volume is

$$\pi\Big((\text{outer radius})^2 - (\text{inner radius})^2\Big) \cdot \text{height}$$

where the height is either dx or dy.

6.2.re2. Let R be the region bounded by the curve $y = 6 - x^2$ and the line $y = 4$. Find the volume of the solid obtained by rotating R about the line $y = -1$. Write your answer as a definite integral, but do not evaluate.

Hint: $y = 6 - x^2$ is a parabola that opens down with vertex at $(0, 6)$, and $y = 4$ is a horizontal line with positive slope.



Slicing vertically and rotating each rectangle about the horizontal line $y = -1$ results in a washer shown above right.

6.2.re3. The region bounded by $y = x^2$ and $y = x + 2$ is rotated about the line $y = 4$. Find the volume swept out by the region.

6.2.re4. The triangle in the xy plane having vertices $(0, 0)$, $(2, 1)$, and $(3, 0)$ is rotated about $x = 0$. Find the resulting volume

Answers

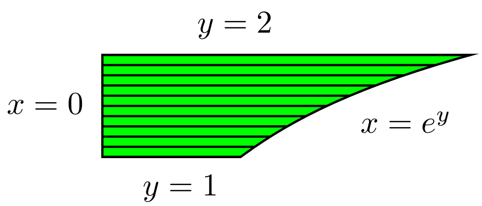
6.2.re1. $\int_0^1 (x - 1 + \cosh x)^2\,dx$ 6.2.re2. $V = \int dV = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\Big((7 - x^2)^2 - 5^2\Big)\,dx$ 6.2.re3. $108\pi/5$
6.2.re4. 5π

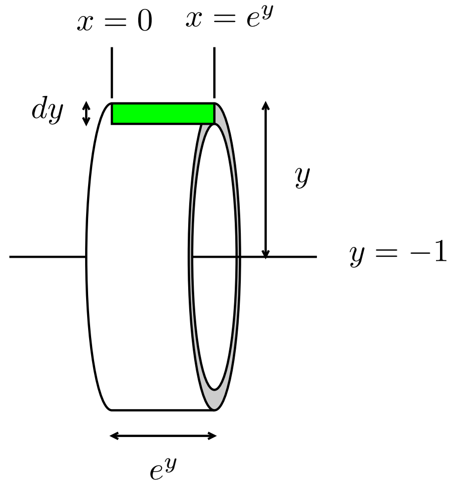
6.3: More Volumes

Volumes of Revolution: the method of cylindrical shells

6.3.re1. The region bounded by the curves $y = \ln x$, $y = 1$, $y = 2$, and $x = 0$, rotated about the line $y = -1$. Find the volume of the resulting solid. Express your answer as a definite integral, but do not evaluate.

Hint:





6.3.re2. Let R be the region in the plane bounded by $y = x$ and $y = 2 - x^2$. Find the volume swept out as R is rotated about the line given line.

a. $x = 4$

b. $y = 2$

Tip: Don't ask "Should I use shells or washers?" Instead, ask "Is it easier to slice the region vertically or horizontally?"

Answers

6.3.re1. $V = \int dV = \int_1^2 2\pi(y+1)e^y \, dy$ 6.3.re2a. $72\pi/5$ 6.3.re2b. $81\pi/2$

6.4: Work

When a constant force is applied to an object moving in a straight line in the same direction as the force, the **work** done by that force is, by definition, its magnitude times the distance traveled by the object.
Work can be measured in foot-pounds or Newton-meters (also known as Joules). For instance, if I lift a 2-pound calculus book 3 feet in the air, I've done 6 foot-pounds of work. The magnitude of the force necessary to lift an object is the object's weight.

6.4.re1. A rope 60 feet long hangs from the top of a tall tower. If the rope weighs 30 lbs, find the work to pull 40 ft of the rope to the top of the tower.

6.4.re2. Suppose a 5-lb weight is attached to the end of the rope in 6.4.re1. Find the work to pull the rope and weight to the top of the tower.

6.4.re3. A one-pound bucket initially holds 50 N of sand, but as it is lifted, the sand leaks from the bucket at a rate of $\frac{1}{2}$ N per m. Find the work to lift the leaky bucket and its contents 100 feet.

Spring problems

A spring has a natural resting length. To *hold* the spring at a length different from its natural length requires a force which has the following simple description.

Hooke's Law. *The force $f(x)$ necessary to hold a spring x meters from its natural length is proportional to x. That is, for some* **spring constant** k,

$$f(x) = kx$$

To *stretch* or *compress* a spring beyond its natural length requires work.

Tips: (1) In a spring-work problem, don't confuse *force*, measured in pounds or Newtons, and *work*, measured in foot-pounds or Joules.
(2) Look for the information that will allow you to solve for the spring constant k.

6.4.re4. A 3-Newton force will hold a spring 0.15m from its resting length.

a. Find the work done in stretching the spring from 0m to 0.2m beyond its natural length.

b. Find the work needed to stretch the spring from 0.2m to 0.3m beyond its natural length.

Tip: All work problems look alike in the following sense: If an object moves along an axis from $x = a$ to $x = b$, and if the force applied to the object when at position x is $f(x)$, then

$$dw = f(x)\,dx \qquad \text{and} \qquad w = \int_a^b f(x)\,dx$$

Answers

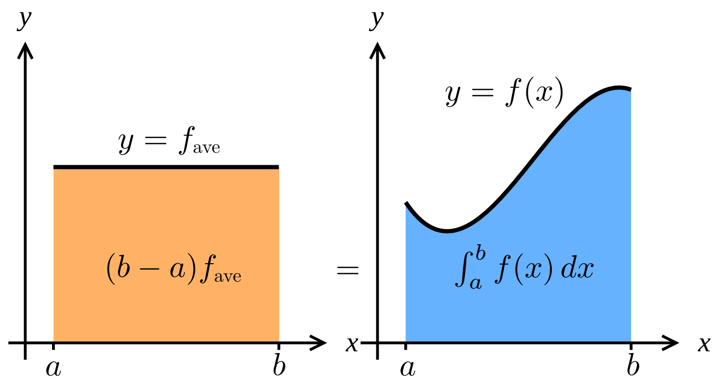
6.4.re1. 800 ft-lbs 6.4.re2. 1000 ft-lbs 6.4.re3. 2600 Nm 6.4.re4a. 2/5 Joule
6.4.re4b. 1/2 Joule

6.5: Average Value

Definition 6.5.re.1. *The* **average value** *of the function $f(x)$ on the interval $[a, b]$ is*

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

f_{ave} is the altitude of the horizontal line that captures the same area as f over $[a, b]$:

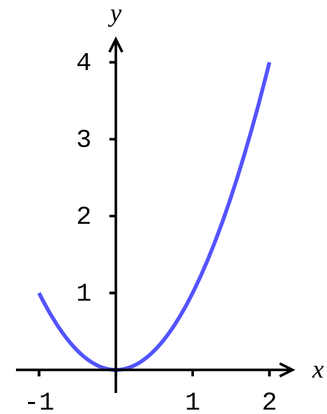


Note that the average value of a given function depends on the interval.

6.5.re1. Calculate the average of x^2 on the given interval.

a. $[-1, 2]$

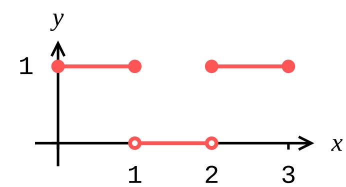
b. $[0, 2]$



6.5.re2. Calculate the average of the function graphed at right on the given interval.

c. $[0, 3]$

d. $[1, 3]$



Mean Value Theorem for Integrals 6.5.re.2. *If $f(x)$ is continuous on the interval $[a, b]$, then there exists at least one number c in $[a, b]$ for which $f(c) = f_{\text{ave}}$.*

6.5.re3. Find the number(s) c in $[-1, 1]$ where x^2 equals its average value on that interval.

6.5.re4. If the velocity of an object at time t seconds was $t(t-2)^2$ m/sec, find the object's average velocity between times $t = 0$ and $t = 2$.

6.5.re5. If the temperature in a greenhouse is $T = 25 + 4\sin\left(\frac{\pi t}{12}\right)$ where t is the number of hours since 6am, compute the average temperature in the greenhouse from noon to 6pm.

Answers

6.5.re1a. 1 6.5.re1b. 4/3 6.5.re2c. 2/3 6.5.re2d. 1/2 6.5.re3. $\pm 1/\sqrt{3}$ 6.5.re4. 4/15 m/sec 6.5.re5. $25 + 8/\pi$

7.1: Integration by Parts

The **Integration by Parts** formula is:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

To integrate $\int f(x)\,dx$ by parts, you must choose u and dv so that:

1. $u\,dv$ equals $f(x)\,dx$ exactly;
2. $\int\, dv$ is an integral you know; and
3. $\int v\,du$ is simpler that the original $\int u\,dv$.

7.1.re1. Integrate:

a. $\int x\sin x\,dx$

b. $\int x^2e^x\,dx$

c. $\int \ln x\,dx$

d. $\int x^3\ln x\,dx$

e. $\int \sin^{-1}\,dx$

f. $\int e^x\cos x, dx$

Reduction formulas

These formulas for trig integrals can all be derived using Integration by Parts, as in Example 6, p. 475 of your textbook. I'll provide these on the exam if they are necessary.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

Tip: When confronted with a new problem, think of past examples that are similar, and what worked in those problems.

Answers

7.1.re1a. $-x\cos x + \sin x + C$ 7.1.re1b. $x^2e^x - 2xe^x + 2e^x + C$ 7.1.re1c. $x\ln x - x + C$
7.1.re1d. $-(x^4/16) + (1/4)x^4\ln x + C$ 7.1.re1e. $\sqrt{1-x^2} + x\sin^{-1}(x) + C$ 7.1.re1f. $\frac{1}{2}e^x(\sin x + \cos x) + C$

7.2: Trigonometric Integrals

You must know the derivatives of the six trig functions (see the table in the differentiation review in these notes) and these trig identities:

PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$
EVEN & ODD IDENTITIES
$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$
SUM & DIFFERENCE IDENTITIES
$\sin(x+y) = \sin x \cos y + \cos x \sin y$ $\cos(x+y) = \cos x \cos y - \sin x \sin y$ $\sin(x-y) = \sin x \cos y - \cos x \sin y$ $\cos(x-y) = \cos x \cos y + \sin x \sin y$
DOUBLE ANGLE FORMULAS
$\sin(2x) = 2\sin x \cos x$ $\cos(2x) = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$
HALF ANGLE FORMULAS
$\cos^2 x = \frac{1}{2}\big(1 + \cos(2x)\big)$ $\sin^2 x = \frac{1}{2}\big(1 - \cos(2x)\big)$

7.2.re1. Derive the 2nd and 3rd Pythagorean identities from the first by dividing by either $\sin^2 x$ or $\cos^2 x$.

7.2.re2. Derive all the sum/difference/double/half-angle formulas from the sum formula for sine by differentiating with respect to x, replacing y with $-y$ and using the even/odd identities, or replacing y with x.

Rule 1: To integrate

$$\int \sin^n x \cos^m x\, dx,$$

if the exponent of $\sin x$ is odd, you can substitute $u = \cos x$;
if the exponent of $\cos x$ is odd, you can substitute $u = \sin x$.

Rule 2: To integrate

$$\int \tan^n x \sec^m x\, dx,$$

if the exponent of $\sec x$ is even, you can substitute $u = \tan x$;
if the exponent of $\tan x$ is odd, you can substitute $u = \sec x$.

Rule 3:
To integrate even powers of tangent and/or odd powers of secant, rewrite the integrand entirely in terms of secant and use the reduction formula for secant.

To integrate even powers of sine and/or cosine, either

a. Rewrite entirely in terms of either sine or cosine and use a reduction formula,

b. use Euler's Formula (as in the next lecture), or

c. use the half-angle identities as in our text (but I don't recommend it).

7.2.re3. Integrate $\tan x\,dx$ and $\int \cot x\,dx$ by rewriting the integrand in terms of sine and cosine.

7.2.re4. Integrate $\sec x$ by first multiplying top and bottom by $(\sec x + \tan x)$. Similarly integrate $\csc x$.

7.2.re5. Integrate:

a. $\int \sin^5 \sqrt{\cos x}\,dx$

b. $\tan^2 x \sec^6 x\,dx$

c. $\tan x \sec^{5/2} x\,dx$

d. $\tan^2 x \sec x\,dx$

For more practice, see
http://kunklet.people.cofc.edu/MATH220/stew0702prob.pdf

Answers

7.2.re3. $\int \tan x, dx = \ln|\sec x| + C$. $\int \cot x\,dx = \ln|\sin x| + C$ 7.2.re4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$ 7.2.re5a. $-\frac{2}{3}\cos^{3/2} x + \frac{4}{7}\cos^{7/2} x - \frac{2}{11}\cos^{11/2} x + C$ 7.2.re5b. $\frac{1}{7}\tan^7 x + \frac{2}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$ 7.2.re5c. $\frac{2}{7}\sec^{7/2} x + C$ 7.2.re5d. $\frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

7.2.5: Trigonometric Integrals using Euler's Formula

See http://kunklet.people.cofc.edu/MATH220/euler.pdf for complete notes.

Euler's formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

implies

$$\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

Euler's formula can be used to rewrite trig functions of the form

$$\sin^{2m} x\cos^{2n} x \qquad \sin(Ax)\sin(Bx) \qquad \sin(Ax)\cos(Bx) \qquad \cos(Ax)\cos(Bx)$$

as sums of **sinusoidal** functions:

$$A\sin(Bx+C)+D \quad \text{or} \quad A\cos(Bx+C)+D$$

(which are easy to integrate).

7.2.5.re1. Use Pascal's triangle to expand the following.

a. $(x+y)^5$ b. $(e^x - e^{-x})^6$ c. $(u^2-1)^4$

7.2.5.re2. Write as a sum of sinusoidal functions.

a. $\cos^4 x$ b. $\sin(3x)\cos(5x)$

Answers

7.2.5.re1a. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$ 7.2.5.re1b. $e^{6x}-6e^{4x}+15e^{2x}-20+15e^{-2x}-6e^{-4x}+e^{-6x}$
7.2.5.re1c. $u^8-4u^6+6u^4-4u^2+1$ 7.2.5.re2a. $\frac{1}{8}(\cos(4x)+4\cos(2x)+3)$ 7.2.5.re2b. $\frac{1}{2}(\sin(8x)-\sin(2x))$

7.3: Trigonometric Substitution

Turn this quadratic	into this trig expression;	you may use
$Ax^2 + B$	$\tan^2\theta + 1$	$\sqrt{\sec^2\theta} = \sec\theta$
$B - Ax^2$	$1 - \sin^2\theta$	$\sqrt{\cos^2\theta} = \cos\theta$
$Ax^2 - B$	$\sec^2\theta - 1$	$\sqrt{\tan^2\theta} = \tan\theta$

7.3.re1. $\displaystyle\int \frac{x^2\,dx}{(x^2+9)^{5/2}}$

7.3.re2. $\displaystyle\int \frac{\sqrt{1-4x^2}}{x^2}\,dx$

7.3.re3. $\displaystyle\int \frac{x^2}{\sqrt{9x^2-4}}\,dx$

Completing the square

When your quadratic is of the form $ax^2 + bx + c$, complete the square and you can make a linear change of variable to eliminate the bx term.

7.3.re4. $\displaystyle\int \frac{1}{(-x^2-4x+5)^{3/2}}\,dx$

Answers

7.3.re1. $\frac{1}{27}\frac{x^3}{(x^2+9)^{3/2}} + C$ 7.3.re2. $-\frac{\sqrt{1-4x^2}}{x} - 2\sin^{-1}(2x) + C$
7.3.re3. $\frac{2}{27}(\frac{3}{4}x\sqrt{9x^2-4} + \ln|3x + \sqrt{9x^2-4}|) + C$ 7.3.re4. $\frac{1}{9}\frac{x+2}{\sqrt{9-(x+2)^2}} + C$

7.4: Partial Fractions

Partial fractions is a technique that allows us to integrate any rational function, provided we can factor the denominator.

The rational function $\dfrac{p(x)}{d(x)}$ is called **proper** if $\deg p < \deg d$. If $\dfrac{p(x)}{d(x)}$ is not proper, use long division to rewrite the rational function in the form

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \text{ with } \deg r(x) < \deg d(x).$$

Then find the Partial Fraction Decomposition (**PFD**) of the proper rational function.

ax^2+bx+c is **irreducible** if it can't be factored over the real numbers. Every polynomial with real coefficients factors uniquely (up to nonzero constant factors) into powers of linear factors $(ax+b)^k$ and of irreducible quadratic $(ax^2+bx+c)^k$ factors.

For each linear factor $(ax+b)^k$ of $d(x)$, the PFD includes

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{D}{(ax+b)^k},$$

For each irreducible quadratic factor $(ax^2+bx+c)^\ell$ of $d(x)$, the PFD includes

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \cdots + \frac{Ex+F}{(ax^2+bx+c)^\ell}.$$

7.4.re1. Find the PDF. Precede with long division if necessary.

a. $\dfrac{2x^2+5x+5}{(x+2)^2(x+1)}$ b. $\dfrac{x^4+x^2+x-18}{x^3-x^2+4x-4}$

There are more partial fraction practice problems at
http://kunklet.people.cofc.edu/MATH220/stew0704prob.pdf.

Here are examples of the kinds of fractions produced by partial fractions. Make sure you can integrate them.

7.4.re2. Integrate.

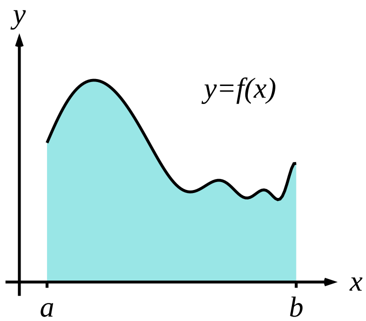
a. $\displaystyle\int \left(\frac{1}{2x-1} - \frac{3}{(2x-1)^2}\right) dx$ b. $\displaystyle\int \frac{x-5}{x^2+4}\,dx$ c. $\displaystyle\int \frac{dx}{x^2+4x+5}$

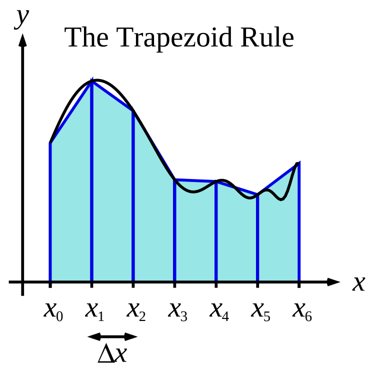
Answers

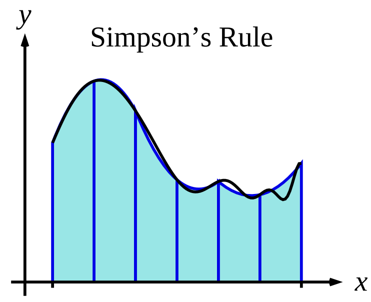
7.4.re1a. $\frac{2}{x+1} - \frac{3}{(x+2)^2}$ 7.4.re1b. $x+1-\frac{3}{x-1}+\frac{x+2}{x^2+4}$ 7.4.re2a. $\frac{1}{2}\ln|2x-1| + \frac{3}{2}(2x-1)^{-1}+C$
7.4.re2b. $\frac{1}{2}\ln(x^2+4) - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$ 7.4.re2c. $\tan^{-1}(x+2)+C$

7.7: Numerical Integration

Numerical integration is the numerical approximation of a definite integral, typically, one that we're unable to evaluate directly.







The **Trapezoid Rule** and **Simpson's Rule** for n subintervals are

$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\big)$$

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)$$

Simpson's Rule requires n to be even. You should know both of these formulas.

7.7.re1. Approximate $\int_0^\pi \sin(x^2)\,dx$ using the trapezoid rule and Simpson's rule with $n = 6$.

$T_6 =$

$$\frac{\pi}{12}\big(\sin 0 + 2\sin(\frac{\pi^2}{36}) + 2\sin(\frac{4\pi^2}{36}) + 2\sin(\frac{9\pi^2}{36}) + 2\sin(\frac{16\pi^2}{36}) + 2\sin(\frac{25\pi^2}{36}) + \sin(\pi^2)\big)$$

and

$S_6 =$

$$\frac{\pi}{18}\big(\sin 0 + 4\sin(\frac{\pi^2}{36}) + 2\sin(\frac{4\pi^2}{36}) + 4\sin(\frac{9\pi^2}{36}) + 2\sin(\frac{16\pi^2}{36}) + 4\sin(\frac{25\pi^2}{36}) + \sin(\pi^2)\big)$$

The **Error** of either of these is defined to be the integral minus the approximation .

$$E_T = \int_a^b f(x)\,dx - T_n \qquad E_S = \int_a^b f(x)\,dx - S_n$$

Fact 7.7.re.1. *If K and L are numbers so that*

$$|f''(x)| \le K \qquad \text{and} \qquad |f^{(4)}(x)| \le L$$

for all x in the interval $[a, b]$, then

$$\boxed{|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4}}$$

I'll provide both of these boxed formulas on exams if they are needed.

These used two ways:

7.7.re2. Suppose $|f^{(2)}(x)| \le 3$ and $|f^{(4)}(x)| \le 16$ on $[10, 20]$.

a. If we approximate $\int_{10}^{20} f(x)\,dx$ using TRAP with $n = 20$, find the largest possible $|E_T|$.

b. Find an upper bound for the error that could occur if we approximate $\int_{10}^{20} f(x)\,dx$ using SIMP and $n = 20$.

c. For what range of n can we be certain that $|E_T| < 10^{-8}$?

d. What values of n will ensure $|E_S|$ is no more than 10^{-8}?

For more error analysis problems, see
http://kunklet.people.cofc.edu/MATH220/stew0707prob.pdf

Answers

7.7.re2a. $|E_T| \le \frac{3\cdot 10^3}{12\cdot 20^2}$ 7.7.re2b. $|E_S| \le \frac{16\cdot 10^5}{180\cdot 20^4}$ 7.7.re2c. $n \ge \sqrt{\frac{3\cdot 10^3}{12}}10^4$
7.7.re2d. $n \ge \sqrt[4]{\frac{16\cdot 10^5}{180}}10^2$

4.4: l'Hospital's Rule and indeterminant forms.

l'Hôpital's Rule 4.4.re.1. *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\pm\infty$, *and if* $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *exists and also equals* L.

Indeterminate forms

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

l'Hospital's Rule applies only to the first two of these. Any others would have to be rewritten as a quotient in order to use l'Hôpital's Rule.

4.4.re1. $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

4.4.re2. $\lim_{x\to\infty} x^{-2}e^x$

4.4.re3. $\lim_{x\to\infty} \left(3x - \sqrt{9x^2-2x}\right)$

When the variable appears in both the exponent and the base, take the limit of the logarithm:

4.4.re4. $\lim_{x\to 0^+} (e^x-1)^x$

4.4.re5. $\lim_{x\to\infty} (x^2+1)^{2/\ln x}$

4.4.re6. $\lim_{x\to 0} (1-2x)^{1/x}$

When l'Hospital's Rule fails to help, try factoring out the dominant terms of the top and bottom.

4.4.re7.

a. $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x}$

b. $\lim_{x\to\infty} \frac{2x}{3x+\sqrt{x^2+x-1}}$

c. $\lim_{x\to\infty} \frac{e^x}{2e^x-e^{-x}}$

d. $\lim_{x\to-\infty} \frac{e^x+2e^{-x}}{2e^x-e^{-x}}$

e. $\lim_{x\to\infty} \tanh x$

f. $\lim_{x\to-\infty} \tanh x$

g. $\lim_{x\to\infty} \frac{\cosh x+2\sinh x}{2\cosh x-\sinh x}$

h. $\lim_{x\to-\infty} \frac{\cosh x+2\sinh x}{2\cosh x-\sinh x}$

Answers

4.4.re1. $\frac{1}{6}$ 4.4.re2. ∞ 4.4.re3. $\frac{1}{3}$ 4.4.re4. 1 4.4.re5. e^4 4.4.re6. e^{-2} 4.4.re7a. -1 4.4.re7b. 1/2 4.4.re7c. 1/2 4.4.re7d. -2 4.4.re7e. 1 4.4.re7f. -1 4.4.re7g. 3 4.4.re7h. $-1/3$

7.8: Improper Integrals

A definite integral $\int_a^b f(x)\,dx$ is said to be **improper** if $f(x)$ has a vertical asymptote somewhere on $[a,b]$, or if a or b is infinite. The Fundamental Theorem of Calculus cannot be applied to an improper integral.

An improper integral is defined to be the limit of proper integrals.

An improper integral is said to **converge** or **diverge**, depending on whether this limit exists or not. x-values where the integrand has a vertical asymptote, or any infinite endpoints of the interval, are **bad points**.

1. Rewrite the integral as a sum, if necessary, so that the bad points are endpoints.
2. Allow only one bad point per integral.
3. When the improper integral has been written as a sum, require each integral in the sum to converge in order for the sum to converge.

7.8.re1. Evaluate the improper integral, if it converges.

a. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$ b. $\displaystyle\int_3^\infty \frac{1}{x^2-x-2}\,dx$ c. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^{1/3}}$

The p-integral

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if } p>1\text{, and diverges to } \infty \text{ if } p\le 1.$$

7.8.re2. $\displaystyle\int_1^\infty \frac{1}{x^{1.001}}\,dx$ (circle one) converges / diverges to ∞.

7.8.re3. $\displaystyle\int_1^\infty \frac{1}{x^{0.999}}\,dx$ (circle one) converges / diverges to ∞.

Sometimes, if its integrand is positive and an improper integral is impossible to evaluate, we can still determine whether or not it converges:

Comparison Test 7.8.re.1. *If $f(x) \geq g(x) \geq 0$ on $[a, b]$, and if $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are both improper, then*

if $\int_a^b f(x)\,dx$ converges, then so must $\int_a^b g(x)\,dx$.

Equivalently,

if $\int_a^b g(x)\,dx$ diverges to infinity, then so must $\int_a^b f(x)\,dx$.

7.8.re4. Determine the convergence or divergence of the improper integral: $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Answers

7.8.re1a. $\frac{4}{3}$ 7.8.re1b. $\frac{1}{3}\ln 4$ 7.8.re1c. diverges 7.8.re2. converges 7.8.re3. diverges to ∞
7.8.re4. converges

8.1: Arclength

$$s = \int ds \qquad \text{where} \quad ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Arclength problems are sometimes carefully prepared to make the integration easy. The trick is to recognize a perfect square under the radical when it's there.

8.1.re1. Find the length of $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ on $1 \leq x \leq 2$.

8.1.re2. Find the length of $y = \ln(x^2 - 1)$ on $2 \leq x \leq 3$.

Answers

8.1.re1. $\frac{5}{3}$ 8.1.re2. $1 + \ln 3 - \ln 2$

8.2: Surface areas of revolution

$$dA = \begin{cases} 2\pi x\, ds & \text{when rotating about the } y\text{-axis} \\ 2\pi y\, ds & \text{when rotating about the } x\text{-axis} \end{cases}$$

Write the integral in terms of a single variable in order to integrate.

8.2.re1. Find the area swept out by $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ for $1 \le x \le 2$ as it is rotated about

a. the y-axis

b. the x-axis.

Answers

8.2.re1a. $\pi(\frac{15}{4} + \ln 2)$ 8.2.re1b. $\frac{47}{16}\pi$

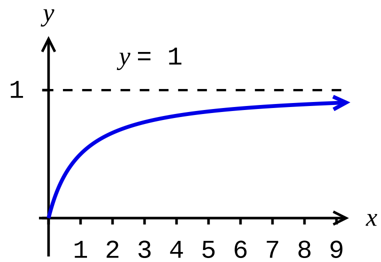
11.1: Sequences and their limits

A **sequence** is a function defined on the integers, usually the nonnegative integers $n \geq 0$ or the positive integers $n > 0$. The big question: what is the $\lim_{n\to\infty} a_n$?

Useful fact 11.1.re.1. *If* $\lim\limits_{x\to\infty} f(x) = L$, *then* $\lim\limits_{n\to\infty} f(n) = L$.

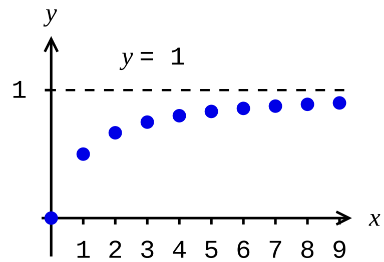
11.1.re1.

The graph of $y = f(x) = \dfrac{x}{x+1}, \quad x \geq 0$



$$\lim_{x\to\infty} \frac{x}{x+1} = 1$$

The graph of $y = a_n = \dfrac{n}{n+1}, \quad n \geq 0$



$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

11.1.re2. Multiple ways of writing the same sequence:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\} \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad \left\{\frac{n}{n+1}\right\} \qquad a_n = \frac{n}{n+1}$$

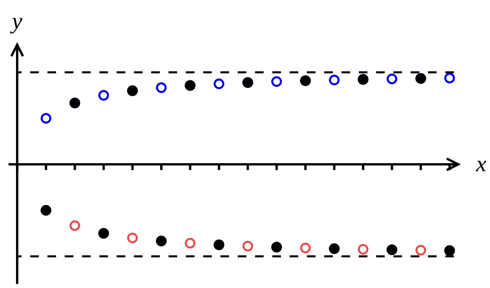
11.1.re3. Find b_n if $\{b_n\} = \left\{0, \frac{1}{3}, -\frac{8}{5}, \frac{27}{7}, -\frac{64}{9}, \dots\right\}$.

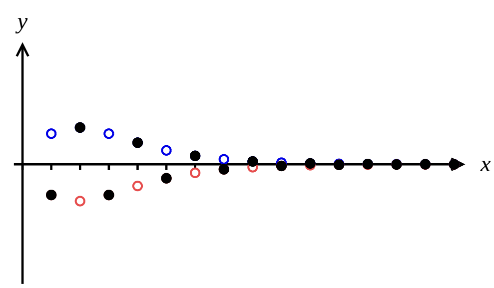
Useful fact 11.1.re.2. $\lim\limits_{n\to\infty} a_n = 0$ *if and only if* $\lim\limits_{n\to\infty} |a_n| = 0$.

11.1.re4.

a. $\left\{\dfrac{(-1)^n n}{n+1}\right\}$

b. $\left\{\dfrac{(-1)^n n}{n^2+1}\right\}$



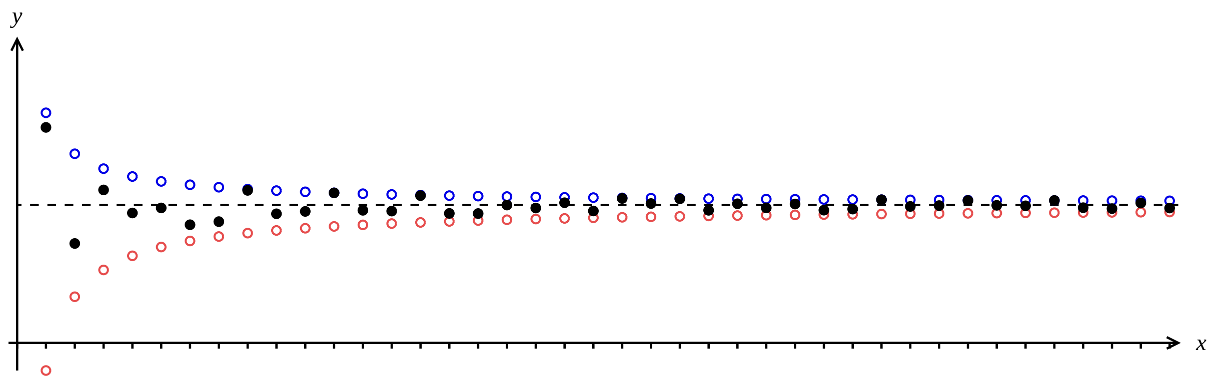


$n!$ (read "n **factorial**") is defined as

$$n! = \begin{cases} n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n > 0, \text{ and} \\ 1 & \text{if } n = 0. \end{cases}$$

$\lim_{n\to\infty} n! = \infty$

Useful fact (squeeze theorem) 11.1.re.3. *If $a_n \le b_n \le c_n$ for all n, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n$ exists and also equals L.*



11.1.re5. Find the limit of the following sequences

a. $n^{-2}\sin(n^2)$

b. $\dfrac{2n + \sin n}{3n + 4}$

Famous Limits Everybody Should Know (FLESK)

http://kunklet.people.cofc.edu/MATH220/flesk.pdf

1. If r is a constant, then $\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1, \\ 1 & \text{if } r = 1, \text{ and} \\ 0 & \text{if } -1 < r < 1, \end{cases}$ but does not exist if $r \le -1$.

2. If c is a positive constant, then $\lim_{n\to\infty} c^{1/n} = 1$.

3. If p is a real constant, then $\lim_{x\to\infty} x^p = \begin{cases} \infty & \text{if } p > 0, \\ 1 & \text{if } p = 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$

4. If k is a real constant, then $\lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^x = e^k$.

5. If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\text{lead term of } p(x)}{\text{lead term of } q(x)}$$

Consequently, $$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} (\text{lead term of } p(x))$$

and $$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} \pm\infty & \text{if } \deg p > \deg q, \\ \dfrac{\text{lead coefficient of } p}{\text{lead coefficient of } q} & \text{if } \deg p = \deg q, \text{ and} \\ 0 & \text{if } \deg p < \deg q. \end{cases}$$

6. If $p(x)$ is a polynomial, then $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1$.

7. If $p(x)$ is a nonzero polynomial, then $\lim_{x\to\infty} |p(x)|^{1/x} = 1$.

11.1.re6. Find the limit of the following sequences

a. $\{3n - \sqrt{9n^2 - 2n}\}$ b. $\left\{\frac{2n^3 + 3n + 1}{5n^3 - 4}\right\}$ c. $\left\{\left(\frac{n+2}{n}\right)^n\right\}$

d. $\{1, 0, -1, 0, 1, 0, -1, \ldots\}$ e. $\left\{\frac{(-1)^n n^2 + 1}{5n^3 - 4}\right\}$

Answers

11.1.re3. $b_n = (-1)^{n+1} \frac{n^3}{2n+1}$ $(n \ge 0)$ 11.1.re5a. 0 11.1.re5b. $\frac{2}{3}$ 11.1.re6a. $\frac{1}{3}$ 11.1.re6b. $\frac{2}{5}$
11.1.re6c. e^2 11.1.re6d. diverges 11.1.re6e. 0

11.2: Series and their sums

If a_i is a sequence, then $\sum_{i=1}^{n} a_i$ stands for the the sum $a_1 + a_2 + \cdots + a_n$.

11.2.re1. Find the sum: $\sum_{n=0}^{4}(-1)^n(n+1)^2$

$\{a_1, a_2, a_3, \ldots\}$ is an **infinite sequence**. Its **limit** denoted $\lim_{i\to\infty} a_i$.

$a_1 + a_2 + a_3 + \cdots$ is an **infinite series**. Its **sum** denoted $\sum_{i=1}^{\infty} a_i$.

$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the nth **partial sum** of the series.

The sum of a series is defined to be the limit of its partial sums:

Definition 11.2.re.1. $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$, *if this limit exists.*

The symbols $\lim\limits_{n\to\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ **do not** mean the same thing:

11.2.re2. Find the following

a. the limit of the sequence $\{1, 0, 0, 0, \ldots\}$ b. the sum of the series $1 + 0 + 0 + 0 + \cdots$
c. $\lim_{n\to\infty}(\frac{1}{2})^n$ d. $\sum_{n=0}^{\infty}(\frac{1}{2})^n$ (see Fact 11.2.re.3 below.)

Definition 11.2.re.2. *A* **geometric series** *is one of the form*

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{j=0}^{\infty} ar^j$$

for some numbers a and r.

Fact 11.2.re.3. *If $a \neq 0$, then $\displaystyle\sum_{j=0}^{\infty} ar^j$ converges if $|r| < 1$, and diverges if $|r| \geq 1$.*

When it converges, $\displaystyle\sum_{j=0}^{\infty} ar^j = \frac{a}{1-r}$.

11.2.re3. Find the sum of the series, if it converges. Express your answer as a rational number.

a. $\dfrac{2}{3} - \dfrac{4}{9} + \dfrac{8}{27} - \dfrac{16}{81} + \cdots$ b. $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{2}{3}\right)^{n-2}$ c. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{2n-2}}{5^{n+1}}$

11.2.re4. Write the repeating decimal $1.\overline{21}$ as a series and express its sum as a rational number.

Definition 11.2.re.4. *A* **telescoping series** *is one of the form* $\sum_{k=0}^{\infty}(b_{k+1} - b_k)$ *for some sequence* b_k*.*

The partial sums of a telescoping series collapse to a few terms. To find a formula for the nth partial sum, find the first few and observe the pattern.

11.2.re5. Find the sum of the series by first finding a formula for its nth partial sum.

a. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Most series other than geometric or telescoping series are difficult to sum, but in the next sections, we'll learn several tests that sometimes allow us to conclude that a series converges or diverges. The first of these is the nth term test:

The nth term test 11.2.re.5. *If* $\lim_{n\to\infty} a_n$ *is not zero, then the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *must diverge.*

11.2.re6. What does the nth term test say about the following series?

a. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ c. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

The converge of the nth term test is false. If the nth term of a series goes to zero, the series might still diverge. That is, the nth term test *sometimes* allows us to conclude that a series diverges. It *never* allows us to conclude that a series converges.

Useful rules 11.2.re.6. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *both converge and if c is any number, then*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \text{ and}$$
$$\sum_{n=1}^{\infty} c \cdot a_n = c\left(\sum_{n=1}^{\infty} a_n\right).$$

However,

$$\sum_{n=1}^{\infty}(a_n b_n) \neq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

Answers

11.2.re1. $1 - 4 + 9 - 16 + 25 = 15$ 11.2.re2a. 0 11.2.re2b. 1 11.2.re2c. 0 11.2.re2d. 2
11.2.re3a. $\frac{2}{5}$ 11.2.re3b. 18 11.2.re3c. diverges 11.2.re4. $\sum_{n=0}^{\infty} \frac{12}{10}\left(\frac{1}{100}\right)^n = \frac{40}{33}$
11.2.re5a. $s_n = \ln(n+1)$. series diverges 11.2.re5b. $s_n = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. sum $= \frac{3}{2}$.
11.2.re6a. series diverges 11.2.re6b. inconclusive 11.2.re6c. inconclusive

11.3: The Integral Test

Fact 11.3.re.1. *If $a_n \geq 0$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ either diverges to ∞ or converges (to a finite sum).*

The integral and comparison tests both reply on this fact, so they both apply only to **nonnegative series**: that is, series with no negative terms.

The Integral Test 11.3.re.2. *If $f(x)$ is positive and decreasing on the interval $[k, \infty)$ for some number k, then*

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_{1}^{\infty} f(x)\,dx$$

either both converge or both diverge.

11.3.re1. The **p-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. The (divergent) p-series with $p = 1$ is called the **harmonic series**. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

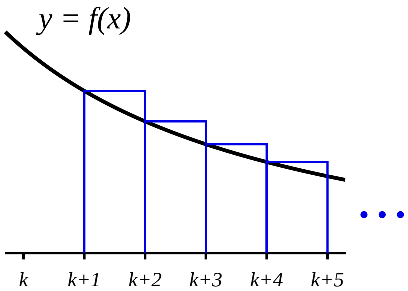
11.3.re2. Determine whether the series converges or diverges.

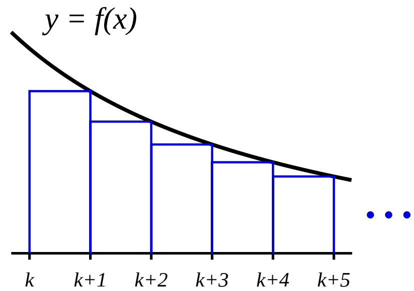
a. $1 + \dfrac{1}{\sqrt[3]{2}} + \dfrac{1}{\sqrt[3]{3}} + \dfrac{1}{\sqrt[3]{4}} + \cdots$ b. $-2 - \dfrac{1}{4} - \dfrac{2}{27} - \dfrac{1}{32} - \dfrac{2}{125} - \dfrac{1}{108} - \cdots$

11.3.re3. Use the integral test to determine the convergence/divergence of the series. Can you think of an easier test to apply?

a. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 - 2}$

The integral test provides a useful error bound:





$$\int_{k+1}^{\infty} f(x)\,dx \le \sum_{n=k+1}^{\infty} f(n) \le \int_{k}^{\infty} f(x)\,dx$$

Note that this provides upper and lower bounds for the **error**

$$\sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{k} f(n) = \sum_{n=k+1}^{\infty} f(n)$$

when we use the kth partial sum to approximate the sum of the series.

11.3.re4. Find an upper bound for the error that occurs when $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^4}$ is approximated by its 10th partial sum $\sum_{n=1}^{10} \frac{1}{n(\ln n)^4}$.

11.3.re4, continued. If I wish for the error to be less than 10^{-6}, how many terms should I use in my approximation?

For more practice problems like11.3.re4, see
http://kunklet.people.cofc.edu/MATH220/stew1103prob.pdf.

Answers

11.3.re2a. $p = 1/3$. diverges. 11.3.re2b. series equals $-2\sum_{n=1}^{\infty} \frac{1}{n^3}$:-) See 11.2.re.6. Series converges.
11.3.re3a. converges, but root test would be easier 11.3.re3b. conveges. Could limit-compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. 11.3.re3c. diverges. could compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. 11.3.re4. Error $< \frac{1}{3(\ln 10)^3}$
11.3.re4. $k > e^{100/\sqrt[3]{3}}$

11.4: The Comparison Test

The Comparison Test 11.4.re.1. *Suppose $0 \le a_n \le b_n$, for all $n \ge k$ for some k.*
If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ must also converge.
If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ must also diverge.

11.4.re1. Determine the convergence/divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} x}{n}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{(n+1)3^n}$ d. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n+n}{2^n}$

If our series and the series we would compare to it aren't related in a way that leads to any conclusion via the Comparison Test, there's another test we might try.

Limit Comparison Test 11.4.re.2. *If a_n and b_n are both ≥ 0 for all n, and if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive (and finite) number, then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

11.4.re2. Determine the convergence/divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n-1}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+\pi}$

Limit Comparison Test, continued.

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $0 \le a_n < b_n$ for all $n > k$ for some number k.

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $a_n > b_n \ge 0$ for all $n > k$ for some number k.

In either of these events, we can then try the ordinary comparison test.

11.4.re3. Determine the convergence/divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ by limit-comparing it to the harmonic series.

Answers

11.4.re1a. converges. compare to $\sum_{n=1}^{\infty} \frac{1/2}{n^2}$. 11.4.re1b. diverges. compare to $\sum_{n=1}^{\infty} \frac{\pi/4}{n}$.
11.4.re1c. converges. compare to $\sum_{n=1}^{\infty} (\frac{2}{3})^n$. 11.4.re1d. diverges. compare to $\sum_{n=3}^{\infty} (\frac{3}{2})^n$.
11.4.re2a. converges. limit-compare to $\sum_{n=1}^{\infty} (\frac{3}{4})^n$. 11.4.re2b. diverges. limit-compare to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.
11.4.re3. Since $\lim_{n\to\infty} \frac{1/\ln n}{1/n} = \infty$ (by l'Hospital's), $\frac{1}{\ln n} > \frac{1}{n} > 0$ eventually. Because the harmonic series diverges, so does $\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

11.5: The Alternating Series Test

An **alternating series** is one that alternates in sign, such as

$$-\frac{1}{2}+\frac{2}{3}-\frac{1}{4}+\frac{4}{9}-\frac{1}{8}+\frac{8}{27}-\cdots \tag{11.5.re.1}$$

Alternating Series Test 11.5.re.2. *If $b_n \geq b_{n+1}$ for all n in $[k,\infty)$ for some number k, and if $\lim_{n\to\infty} b_n = 0$, then the alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ converges.*

Note that $\sum_{n=0}^{\infty}(-1)^{n+1}b_n = -\sum_{n=0}^{\infty}(-1)^n b_n$, so if either converges, so does the other.

> If $\lim_{n\to\infty} b_n$ fails to be zero, by Useful Fact 11.1.2 (lecture notes), $\lim_{n\to\infty}(-1)^n b_n$ cannot equal zero, and therefore $\sum_{n=0}^{\infty}(-1)^n b_n$ must diverge by the nth Term Test.

If b_n fails to be decreasing, then the Alternating Series Test is **inconclusive**.

11.5.re1. Because $\frac{1}{n}$ is decreasing and goes to zero as $n\to\infty$, the **alternating harmonic series** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges by the Alternating Series Test.

11.5.re2. Determine the convergence/divergence of the series.

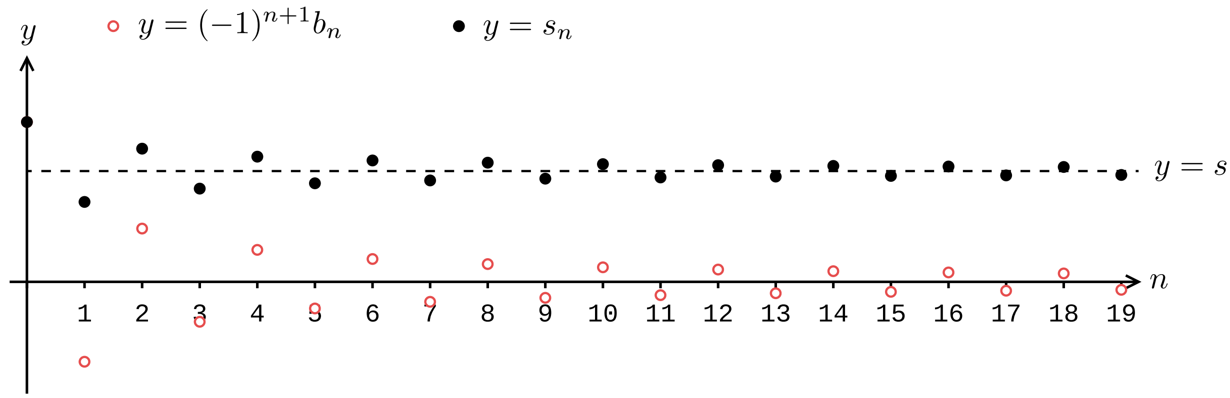
a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+20}$

b. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}(e^{1/n}-1)$

c. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{e^{-n}+1}{1-e^{-n}}$

d. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n-1}ne^{-n/9}$

Error analysis in the alternating series test



Alternating Series Test, continued. *Under the same hypotheses as the Alternating Series Test, the sum of the series lies between any two consecutive partial sums. Consequently, the absolute error* $|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$.

11.5.re3. Suppose we approximate $s = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3+1}$ by the partial sum of its first 20 terms.

$$s \approx s_{20} = \frac{1}{2} - \frac{1}{9} + \frac{1}{28} - \cdots - \frac{1}{20^3+1}.$$

Find an upper bound on the error of this approximation.

11.5.re3, continued. How many terms of this series must I add to be certain that the partial sum is within 10^{-6} of the sum of the series?

For more practice problems like 11.5.re3, see http://kunklet.people.cofc.edu/MATH220/stew1105prob.pdf.

Answers

11.5.re2a. converges 11.5.re2b. converges 11.5.re2c. diverges by nth term test.
11.5.re2d. converges by Alternating Series Test. could also use root test. 11.5.re3. $|\text{error}| \leq \frac{1}{21^3+1}$
11.5.re3. number of terms $> \sqrt[3]{999,999} - 1$

11.6: Absolute Convergence and the Root and Ratio Tests

Fact 11.6.re.1. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ must also converge.*

Definition 11.6.re.2.

$\sum_{n=1}^{\infty} a_n$ *is* **absolutely convergent** *if* $\sum_{n=1}^{\infty} |a_n|$ *is convergent.*
$\sum_{n=1}^{\infty} a_n$ *is* **conditionally convergent** *if* $\sum_{n=1}^{\infty} a_n$ *is convergent, but* $\sum_{n=1}^{\infty} |a_n|$ *is divergent.*

11.6.re1. Give an example of an absolutely convergent series and a conditionally convergent series. (There are infinitely many correct answers.)

The Root and Ratio Tests 11.6.re.3. *Let L be the following limit, if it exists.*

ROOT TEST	RATIO TEST
$L = \lim_{n\to\infty} \sqrt[n]{\vert a_n\vert}$	$L = \lim_{n\to\infty} \frac{\vert a_{n+1}\vert}{\vert a_n\vert}$

Then, regardless which of these you used to compute L,

if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and
if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges (in fact, $\lim_{n\to\infty} |a_n| = \infty$).

The Root and Ratio Tests are inconclusive if $L = 1$.

The Root and Ratio tests will work poorly on a series that resembles a p-series but will work well on a series that resembles a geometric series. Ratio works well on a series whose n term includes a factorial factor.

11.6.re2. Determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{\sqrt{n+1}}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1+n+n^2}{4^n}\right)$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin n}{2^n+1}$

d. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{n^2+1}$

Answers

11.6.re1. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ is conditionally convergent because it converges by the Alternating Series Test, but $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.
11.6.re2a. diverges by nth term test 11.6.re2b. absolutely convergent by root or ratio
11.6.re2c. absolutely convergent. Compare to $\sum_{n=0}^{\infty} \frac{2}{2^n}$. 11.6.re2d. conditionally convergent.

11.7: Strategy for Testing Series

See the text for a helpful list of tips and examples of how to choose a convergence test for an infinite series.

11.7.re1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{10^{2n}}{(n+1)!}$ c. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+1}\right)^n$ d. $\displaystyle\sum_{n=1}^{\infty} ne^n$

e. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ f. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n - 4^n}$ g. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$ h. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^4 - 2n^3 + 2}$

Answers

11.7.re1. See the video lecture for 11.7 for solutions to these problems.

11.8: Introduction to Power Series

Definition 11.8.re.1. *A* **power series** *in x* **centered at** *a is a series of the form*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Its **coefficients** *are the numbers $c_0, c_1, c_2, \ldots$.*

The nth coefficient c_n may depend on n but not on x.
To say that the function $f(x)$ is the sum of a power series means that $f(x)$ is the limit of the polynomials which are its partial sums.

11.8.re1. The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

converges to $\frac{1}{1-x}$ if $-1 < x < 1$, meaning $\frac{1}{1-x}$ is the limit of these polynomials:

$$\begin{aligned}
&1\\
&1+x\\
&1+x+x^2\\
&1+x+x^2+x^3\\
&\vdots
\end{aligned}$$

You can observe the convergence of these polynomials to $\frac{1}{1-x}$ on $(-1,1)$ at
http://kunklet.people.cofc.edu/MATH220/geopsums.pdf

Theorem 11.8.re.2. *For any power series centered at $x = a$, exactly one of the following is true:*

1. *The series converges only at $x = a$.*
2. *The series converges for all real x.*
3. *There is a number $R > 0$ with the property that the series converges absolutely if $|x - a| < R$, and diverges if $|x - a| > R$.*

In case 3., the power series converges on the interval $(a - R, a + R)$, and can converge absolutely, converge conditionally, or diverge at the endpoints, where $x - a = \pm R$.
The set of points at which a power series converges is called its **Interval of Convergence**, and the half-length of that interval is called its **Radius of Convergence**. Theorem 11.8.re.2 tells us that the center of a power series is always the center of its interval of convergence.

11.8.re2. What is the radius of convergence in cases 1, 2, and 3 in the theorem above?

11.8.re3. Find the interval and radius of convergence of the power series.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n(n^2+1)}$ b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n}$ c. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n+1}$

d. $\displaystyle\sum_{n=0}^{\infty} n!(x-9)^n$ e. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

You can see some partials sums of the series in 11.8.re3.e at
http://kunklet.people.cofc.edu/MATH220/taylormystery.pdf

11.8.re4. Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = 5$ and diverges at $x = -8$. What can you say about the convergence or divergence of these series?

Choose the appropriate response for each part: **Converges Diverges Not enough information**

a. $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 5^n$ b. $\displaystyle\sum_{n=0}^{\infty} c_n 9^n$ c. $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n$ d. $\displaystyle\sum_{n=0}^{\infty} c_n 7^n$

Answers

11.8.re2. 0 in case 1. ∞ in case 2. R in case 3. 11.8.re3a. IOC $= [-5, -1]$. ROC $= 2$.
11.8.re3b. IOC $= (1, 3]$. ROC $= 1$. 11.8.re3c. IOC $= (-\sqrt{3}, \sqrt{3})$. ROC $= \sqrt{3}$.
11.8.re3d. IOC $= \{9\}$. ROC $= 0$. 11.8.re3e. IOC $= (-\infty, \infty)$. ROC $= \infty$.
11.8.re4. From the given information, $5 \le ROC \le 8$. 11.8.re4a. not enough information
11.8.re4b. diverges 11.8.re4c. converges 11.8.re4d. not enough information

11.9: Obtaining Power Series Representations of Given Functions

Obtaining one power series from another by algebraic manipulation

11.9.re1. Find a power series representation of the given function and its interval of convergence.

a. $\dfrac{1}{1+2x}$ b. $\dfrac{x^2}{3+x}$ c. $\dfrac{1}{x^2-9}$ d. $\dfrac{1-2x}{1-4x^2}$

Obtaining one power series from another by differentiation and integration

Theorem 11.9.re.1. *One can differentiate and integrate the sum of a power series term-by-term (provided it converges at more than just one point). Doing so produces a series with the same radius of convergence as the original.*

11.9.re2. Find a power series representation of the given function and its radius of convergence.

a. $\dfrac{1}{(2-x)^2}$ b. $\dfrac{x}{(1+x^2)^3}$ c. $x\ln(1-x^2)$ d. $\tan^{-1}(2x)$

e. $\displaystyle\int \frac{2x\,dx}{1+x^4}$ f. $\tan^{-1}(x^2)$

Answers

11.9.re1a. $\sum_{n=0}^{\infty}(-1)^n 2^n x^n$; IOC $= (-\frac{1}{2}, \frac{1}{2})$. 11.9.re1b. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+2}}{3^{n+1}}$; IOC $= (-3, 3)$. 11.9.re1c. $-\sum_{n=0}^{\infty} \frac{x^{2n}}{9^{n+1}}$; IOC $= (-3, 3)$. 11.9.re1d. same as a. 11.9.re2a. $\sum_{n=1}^{\infty} n\frac{x^{n-1}}{2^{n+1}}$; ROC $= 2$. 11.9.re2b. $\sum_{n=2}^{\infty}(-1)^n \frac{n(n-1)}{2} x^{2n-3}$; ROC $= 1$. 11.9.re2c. $-\sum_{n=0}^{\infty} \frac{x^{2n+3}}{n+1}$; ROC $= 1$. 11.9.re2d. $\sum_{n=0}^{\infty}(-1)^n 2^{2n+1} \frac{x^{2n+1}}{2n+1}$; ROC $= 1/2$. 11.9.re2e. $C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$; ROC $= 1$. 11.9.re2f. same as e.

11.10: Taylor Series

Definition 11.10.re.1. *If $f(x)$ possesses derivatives of all orders at $x = a$,* **Taylor series** *for $f(x)$ centered at a is*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

> The Taylor series for f at a is defined so as to be the only power series centered at a that could possibly sum to $f(x)$.

The **Maclaurin series** *for the function $f(x)$ is the Taylor series for $f(x)$ centered at 0:*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$$

The **binomial series** *is the Maclaurin series for $(1+x)^k$:*

$$\sum_{n=0}^{\infty} \binom{k}{n} x^n,$$

where $\binom{k}{n}$ is the **binomial coefficient***:*

$$\binom{k}{n} = \begin{cases} 1 & \text{if } n = 0, \text{ and} \\ \dfrac{\overbrace{k(k-1)(k-2)\cdots}^{n \text{ factors}}}{\underbrace{n(n-1)(n-2)\cdots}_{n \text{ factors}}} & \text{if } n \neq 0 \end{cases}$$

If k is a nonnegative integer, then $\binom{k}{0}\ \binom{k}{1}\ \binom{k}{2}\ \ldots\ \binom{k}{n}$ is the kth row of Pascal's triangle, and the binomial series is simply the expansion of $(1+x)^k$ according to the binomial theorem. See Section 0.1 of these notes.

11.10.re1. Compute the binomial coefficients.

a. $\binom{5}{3}$ b. $\binom{4}{5}$ c. $\binom{\pi}{3}$ d. $\binom{-1/2}{4}$

Fact 11.10.re.2. *The radius of convergence of the binomial series $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ is*

$$\begin{cases} \infty & \text{if } k \text{ is a nonnegative integer, and} \\ 1 & \text{otherwise.} \end{cases}$$

Fact 11.10.re.3. *Each of the following functions is the sum of its Maclaurin series (within its interval of convergence).*

$$\begin{array}{llll} \dfrac{1}{1-x} & \ln(1-x) & \tan^{-1} x & \sin^{-1} x \\ \sin x & \cos x & e^x & (1+x)^k \end{array}$$

The Maclaurin series for $\sin^{-1} x$ is derived in exercise 11.10.51. The others are collected in Table 1, p. 768. The best way to learn these series is to practice deriving them.

11.10.re2. Find the Maclaurin series for the each of the functions listed above. Don't simply *copy* the series from the book. Instead, *derive* them yourself from known facts and definitions.

11.10.re3. Find the Taylor series for the given function centered at a.

a. $e^x \quad (a = -1)$ b. $\cos(x) \quad (a = \pi/2)$ c. $2x^3 + 3x^2 - 4 \quad (a = 1)$

d. $\ln(1 - x^3) \quad (a = 0)$ e. $(1 + 2x)^{1/2} \quad (a = 0)$ f. $(2 - x)^4 \quad (a = 0)$

Answers

11.10.re1a. 10 11.10.re1b. 0 11.10.re1c. $\frac{\pi(\pi-1)(\pi-2)}{6}$ 11.10.re1d. $\frac{35}{128}$
11.10.re3a. $\sum_{n=0}^{\infty} e^{-1}(x+1)^n/n!$ 11.10.re3b. $\sum_{n=0}^{\infty}(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}/(2n+1)!$
11.10.re3c. $1 + 12(x-1) + 9(x-1)^2 + 2(x-1)^3$ 11.10.re3d. $-\sum_{n=0}^{\infty} x^{3n+3}/(n+1)$, or $-\sum_{n=1}^{\infty} x^{3n}/n$
11.10.re3e. $\sum_{n=0}^{\infty} \binom{1/2}{n} 2^n x^n$ 11.10.re3f. $16 - 32x + 24x^2 - 8x^3 + x^4$

11.11: Taylor Polynomials and Taylor's Theorem

Definition 11.11.re.1. *The nth* **Taylor polynomial** *centered at a for $f(x)$ is the partial sum of its Taylor series centered at a, up to and including the degree-n term:*

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k. \tag{11.11.re.2}$$

Compare with Definition 11.10.re.1.
You can find a Taylor polynomial either by using (11.11.re.2), or, if you happen to know the corresponding Taylor series, by taking its partial sum.

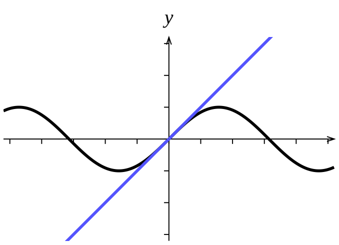
11.11.re1. Find $T_n(x)$ for the given n, function $f(x)$, and center a.

a. e^{-x}, $a=0$, $n=3$ b. $\sin x$, $a=0$, $n=5$ c. $\cos(x^2)$, $a=0$, $n=4$
d. $\tan^{-1} x$, $a=0$, $n=3$ e. $(1-x)^{-1}$, $a=0$, $n=3$ f. $e^x \cos x$, $a=\pi$, $n=3$
g. $x^{3/2}$, $a=4$, $n=3$ h. $\cosh x$, $a=0$, $n=2$ i. $\cosh x$, $a=0$, $n=3$

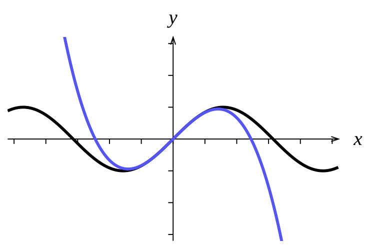
Fact 11.11.re.3. *T_n and its first n derivatives match f and its first n derivatives at $x=a$:*

$$T_n^{(k)}(a) = f^{(k)}(a), \qquad \textit{if } k \le n.$$

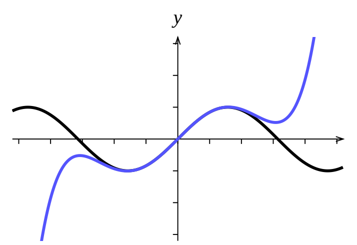
Consequently, the graph of T_n tangent to the graph of f when $n=1$, and even "more tangent" when $n>1$.
For instance, here are the graphs of $\sin x$ and its Maclauren polynomials $T_1(x)$, $T_3(x)$, and $T_5(x)$.



$y = \sin x$
$y = x$



$y = \sin x$
$y = x - \frac{1}{3!}x^3$



$y = \sin x$
$y = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

As these figures suggest, the Taylor polynomials centered at 0 for $\sin x$ are approaching $\sin x$ as $n \to \infty$, and this is true of a great many familiar functions. For example, see Fact 11.10.re.3.
The difference between $f(x)$ and its Taylor Polynomial $T_n(x)$ is called the nth **remainder** of the Taylor series:

$$R_n(x) = f(x) - T_n(x).$$

For $f(x)$ to be the sum of its Taylor series is equivalent to $R_n(x) \to 0$ as $n \to \infty$.

Taylor's Theorem 11.11.re.4. *If f and all of its derivatives exist in an interval containing a and x, then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

Taylor's Inequality 11.11.re.5. *If $|f^{(n+1)}(x)| \le M$ for all x in some interval centered at a, then for all such x,*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Error analysis with Taylor's Theorem

Taylor's Theorem (or Taylor's Inqeuality) is useful when we want to bound the absolute error between f and T_n on an interval about the center a:

$$|f(x) - T_n(x)| \le B \quad \text{for all } x \text{ in} \quad [a-b, a+b]$$

Typical problems give us two of $\{n, b, B\}$ and ask for the third.

11.11.re2. Find an upper bound for $|R_n(x)|$ for the given $f(x)$, n, and a on the given interval.

a. e^{-x}, $a = 1$, $n = 2$, $[0.9, 1.1]$
b. $\sin x$, $a = 0$, $n = 2$, $[-0.5, 0.5]$
c. $\cos x$, $a = 0$, $n = 2$, $[-0.5, 0.5]$
d. $x^{1/2}$, $a = 0$, $n = 3$, $[-0.2, 0.2]$
e. $(x+1)^{3/2}$, $a = 1$, $n = 3$, $[1, 1.2]$
f. $x^2 \ln x$, $a = 1$, $n = 3$, $[0.8, 1.2]$

11.11.re3. Let $f(x) = \sin x$ and $a = 0$. Find b so that the absolute error between f and T_3 is at most 10^{-6} on $[-b, b]$. Tip: 1) In this case, $T_3(x) = T_4(x)$. 2) In problems like this, the larger b, the better the answer.

11.11.re4. What degree Maclaurin polynomial approximates $\cos x$ on the interval $[-1, 1]$ with an absolute error at most 10^{-5}? See table. In problems like this, the smaller n, the better the answer.

n	$1/n!$
5	$8.33E-03$
6	$1.39E-03$
7	$1.98E-04$
8	$2.48E-05$
9	$2.76E-06$
10	$2.76E-07$

Answers

11.11.re1a. $1-x+x^2/2-x^3/6$ 11.11.re1b. $x-x^3/6+x^5/120$ 11.11.re1c. $1-x^4/2$
11.11.re1d. $x-x^3/3$ 11.11.re1e. $1+x+x^2+x^3$ 11.11.re1f. $-e^{\pi}(1+(x-\pi)-(x-\pi)^3/3)$
11.11.re1g. $2^3+\binom{3/2}{1}2(x-4)+\binom{3/2}{3}2^{-1}(x-4)^2+\binom{3/2}{3}2^{-3}(x-4)^3$, or $8+3(x-4)+3(x-4)^2/16-(x-4)^3/128$
11.11.re1h. $1+x^2$ 11.11.re1i. same as h. 11.11.re2a. $\frac{1}{6}e^{-0.9}10^{-3}$ 11.11.re2b. $\frac{1}{48}$
11.11.re2c. $\frac{1}{384}$ 11.11.re2d. $\frac{\sqrt{5}}{128}$ 11.11.re2e. $\frac{3}{5^4 2^{19/2}}$ 11.11.re2f. $\frac{1}{5^2 2^6 3}$ 11.11.re3. $b=\frac{1}{10}\sqrt[5]{12}$
11.11.re4. minimum degree = 8.

9.3: Seperable Differential Equations

Definition 9.3.re.1. *A (1st order)* **differential equation** *is an equation in x, y, and $\frac{dy}{dx}$.*

The **general solution** *of a differential equation is the family of all functions y that satisfy the differential equation, or, more generally, the family of all equations in x and y along which the differential equation is satisfied.*

An **initial value problem** *is a differential equation plus a point (x_0, y_0). Its solution is the particular member of this family that passes through the given point.*

A **separable** differential equation is one which can be written

$$f(x)\,dx = g(y)\,dy$$

To solve a separable equation,

1. **Separate variables**
2. **Integrate**

9.3.re1. Solve the differential equation.

Note: to "solve" a differential equation always means to find its general solution.

a. $\dfrac{dy}{dx} + xy^2 = 0$ b. $\dfrac{dy}{dx} = 1.5y$ c. $\dfrac{1}{y}\dfrac{dy}{dx} = -\dfrac{1}{5}$

d. $x\dfrac{dy}{dx} + \dfrac{\ln x}{y} = 0$ e. $\dfrac{dy}{dx} = \dfrac{(x+1)(y+1)}{x^2y^2}$ f. $e^x\dfrac{dy}{dx} + 2 = y(y-1)$

g. $xe^{x+2y} = \dfrac{dy}{dx}$ h. $\dfrac{du}{dt} = \dfrac{ut^4+u}{t^2+u^3t^2}$ i. $\dfrac{dy}{dx} + x^2\sqrt{1-x^3} = 0$

9.3.re2. Solve the initial value problem.

a. $\dfrac{dy}{dx} = 1.5y$; $y(0) = 4$ b. $x\dfrac{dy}{dx} + \dfrac{\ln x}{y} = 0$; $y = 2$ at $x = 1$

c. $xe^{x+2y} = \dfrac{dy}{dx}$; curve passes through $(2, 0)$

Orthogonal Trajectories

To find the orthogonal trajectories of a family of curves, first use implicit differentiation to find $\frac{dy}{dx}$ along the given family. Then set $\frac{dy}{dx}$ equal the negative reciprocal of this and solve the differential equation.

9.3.re3. Find the orthogonal trajectories of the curves $x^2 + y^3 = C$.

For more orthogonal trajectories, see problem 1 on
http://kunklet.people.cofc.edu/MATH220/stew0903prob.pdf

Answers

9.3.re1a. $y^{-1} = \frac{1}{2}x^2 + C$ 9.3.re1b. $y = y_0e^{1.5x}$ 9.3.re1c. $y = y_0e^{-x/5}$ 9.3.re1d. $-\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$, or $-y^2 = (\ln x)^2 + C$ 9.3.re1e. $\frac{1}{2}y^2 - y + \ln|y+1| = \ln|x| - \frac{1}{x} + C$ 9.3.re1f. $\frac{1}{3}\ln\left|\frac{y-2}{y+1}\right| = -e^{-x} + C$ 9.3.re1g. $(x-1)e^x = -\frac{1}{2}e^{-2y} + C$ 9.3.re1h. $\frac{1}{3}u^3 + \ln|u| = \frac{1}{3}t^3 - t^{-1} + C$ 9.3.re1i. $y = \frac{2}{9}(1-x^3)^{3/2} + C$ 9.3.re2a. $y = 4e^{1.5x}$ 9.3.re2b. $-y^2 = (\ln x)^2 - 4$ 9.3.re2c. $(x-1)e^x = -\frac{1}{2}e^{-2y} + e^2 + \frac{1}{2}$ 9.3.re3. On given family, $\frac{dy}{dx} = \frac{-2x}{3y^2}$, so orthogonal trajectories are the solution to $\frac{dy}{dx} = \frac{3y^2}{2x}$. Solve to obtain $\frac{1}{3y} + \frac{1}{2}\ln|x| = C$.

10.1: Parametric Equations of Curves

A parametric representation of a curve is a pair of equations

$$x = x(t) \qquad y = y(t)$$

so that the curve is the path traced out by a particle whose position at time t is $\big(x(), y(t)\big)$.

10.1.re1. $x = \cos(t)$, $y = \sin(t)$ is the standard parametrization of the unit circle

Deriving an xy-equation of a curve from its parametric representation—if that's possible—is referred to as **elimination of the parameter**.

10.1.re2. Sketch the parametrized curve, indicating the direction the curve is traced with arrows. Eliminate the parameter.

a. $x = -\sin t$, $y = \cos t$

b. $x = \cos(2t)$, $y = -\sin(2t)$

c. $x = 2 + 5\sin t$, $y = 3 - \cos t$

d. $x = \sinh t$, $y = \cosh t$

e. $x = \sin t$, $y = \cos(2t)$

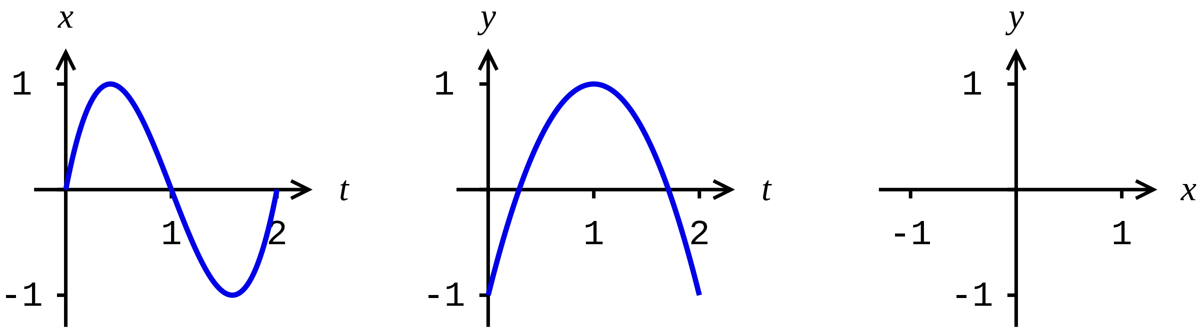
f. $x = t^2 + 1$, $y = -t^2$

g. $x = 2t + 1$, $y = 4 - 3t$

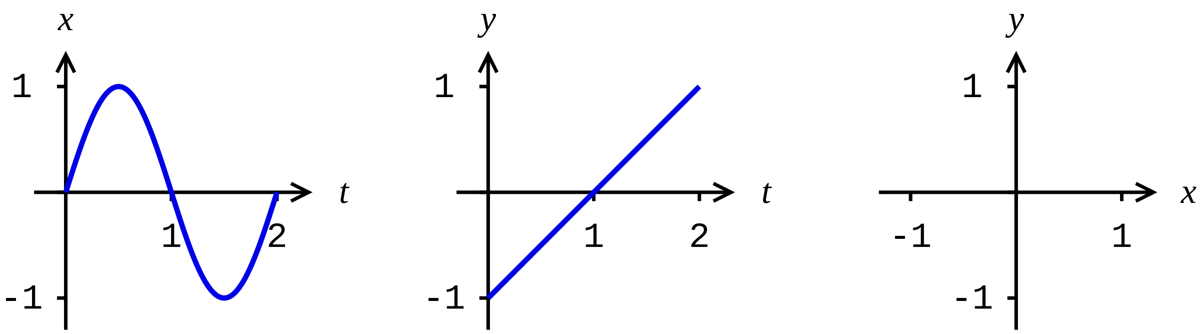
h. $x = \sec^2$, $y = \tan t$, $-\pi/2 < t < \pi/2$

10.1.re3. Use the graphs of $x(t)$ and $y(t)$ to sketch the curve given parametrically by $x = x(t)$, $y = y(t)$. Indicate direction with arrows.

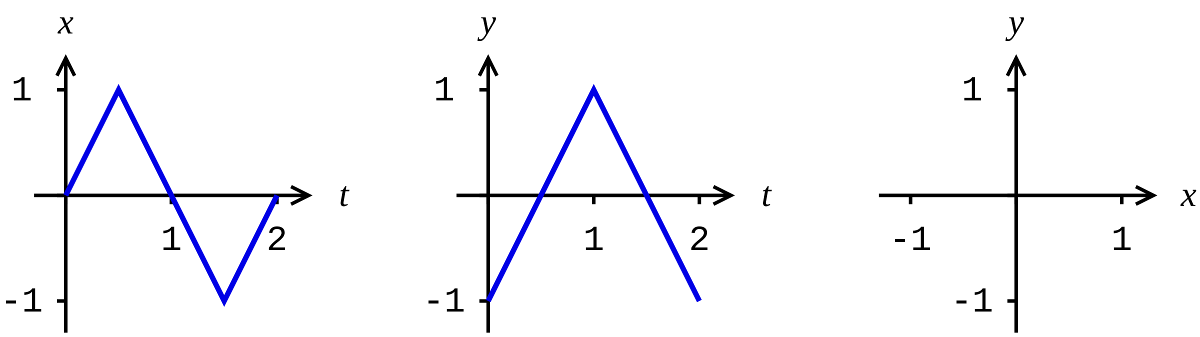
a.



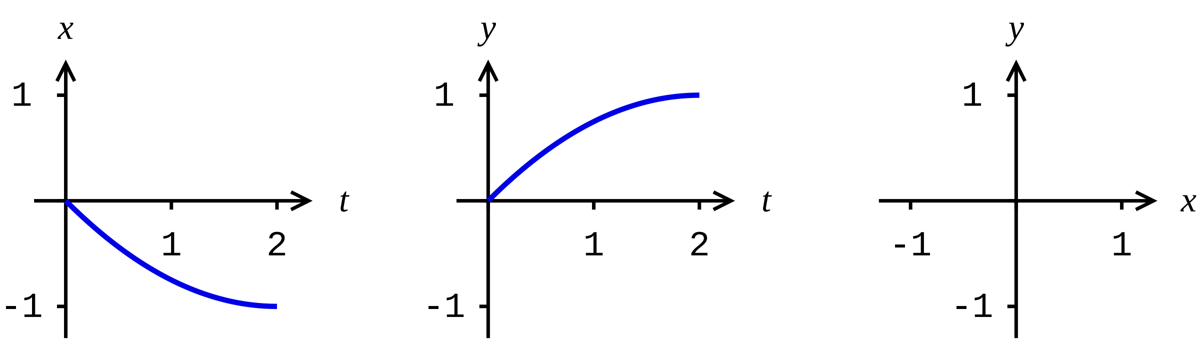
b.



c.



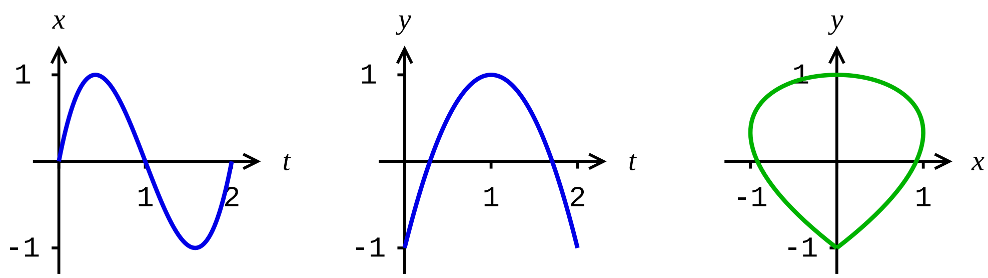
d.



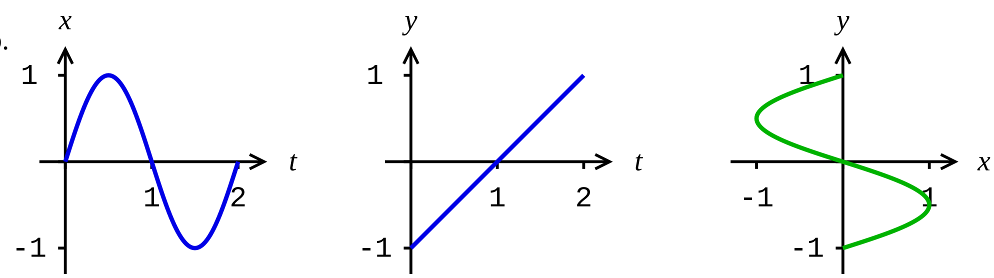
Answers

10.1.re2. See graphs on Desmos.com 10.1.re2a. $x^2 + y^2 = 1$. The unit circle, traced counterclockwise 10.1.re2b. $x^2 + y^2 = 1$. Unit circle, traced clockwise 10.1.re2c. $\frac{1}{25}(x-2)^2 + (y-3)^2 = 1$. Ellipse centered at $(2, 3)$, radius 5 in the x direction and 1 in the y direction, traversed counterclockwise.
10.1.re2d. $y^2 - x^2 = 1$. Upper half of this hyperbola, drawn left to right 10.1.re2e. $1 - 2x^2 = y$. The part of the parabola where $-1 \le x, y \le 1$, redrawn every 2π units of t.) 10.1.re2f. $x + y = 1$. The part of this line where $x \ge 1$, drawn right to left when $t \le 0$ and left to right when $t \ge 0$. 10.1.re2g. The line $y = 4 - \frac{3}{2}(x-1)$, or $3x + 2y = 11$. Drawn left to right. 10.1.re2h. $1 + y^2 = x$, drawn from bottom to top. 10.1.re3. See below.

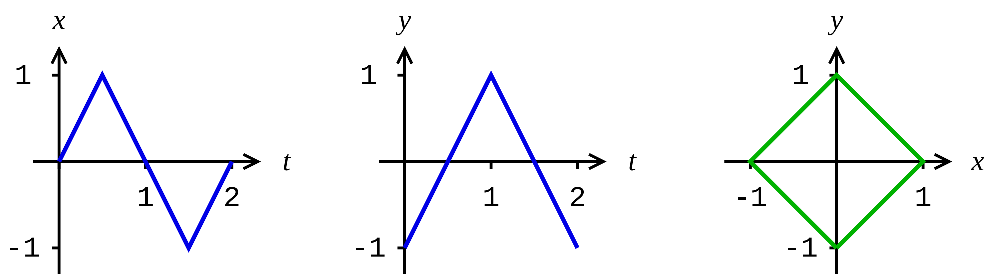
a.



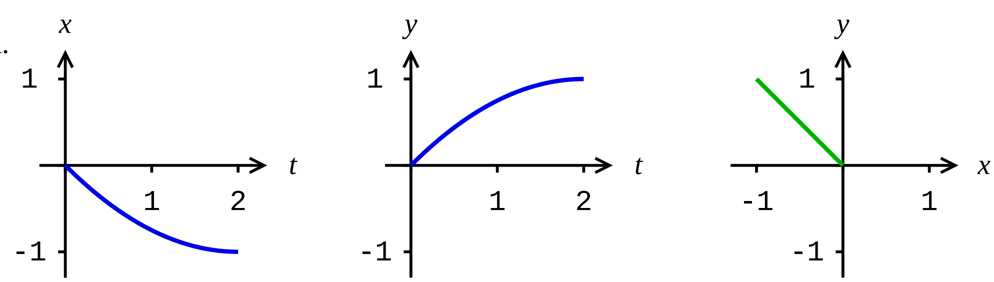
b.



c.



d.



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d. $x = \sinh t$, $y = \cosh t$

e. $x = \sin t$, $y = \cos(2t)$

f. $x = t^2 + 1$, $y = -t^2$

g. $x = 2t + 1$, $y = 4 - 3t$

h. $x = \sec^2$, $y = \tan t$, $-\pi/2 < t < \pi/2$

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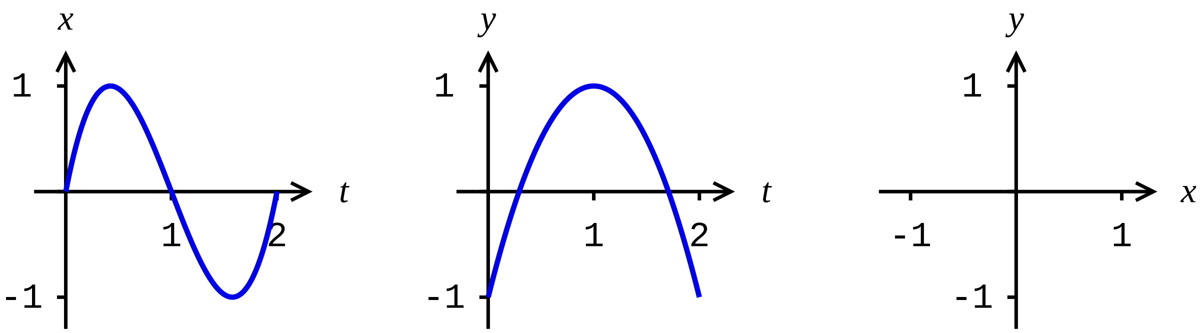
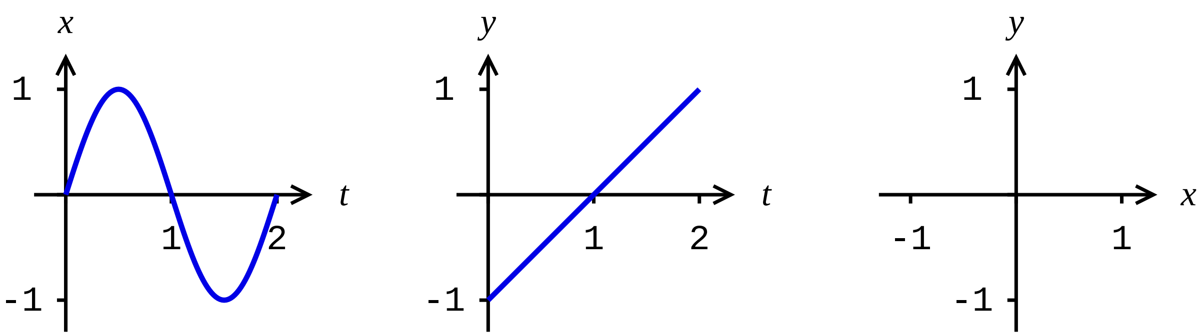
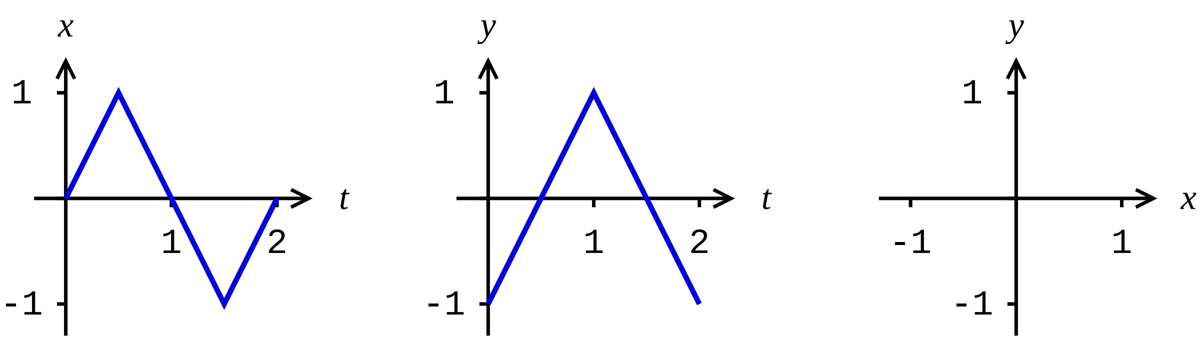
a.



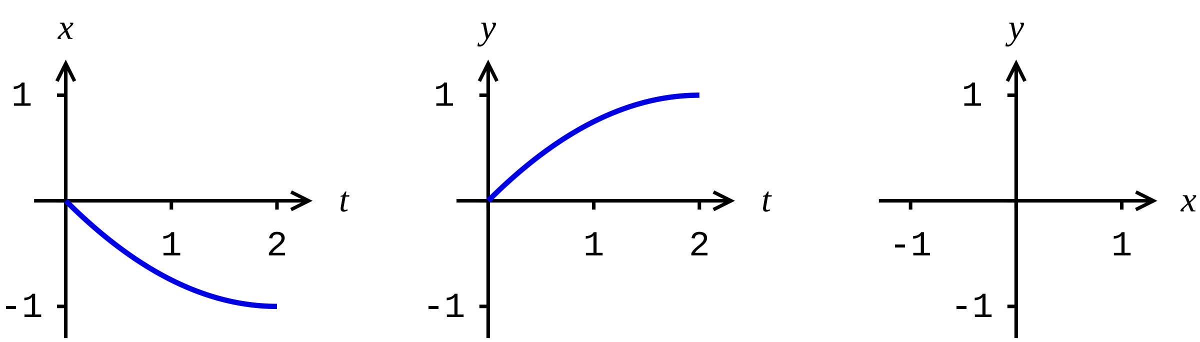
b.



c.



d.



Answers

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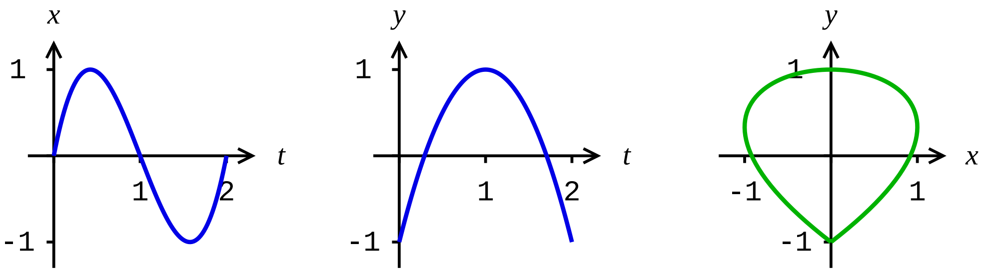
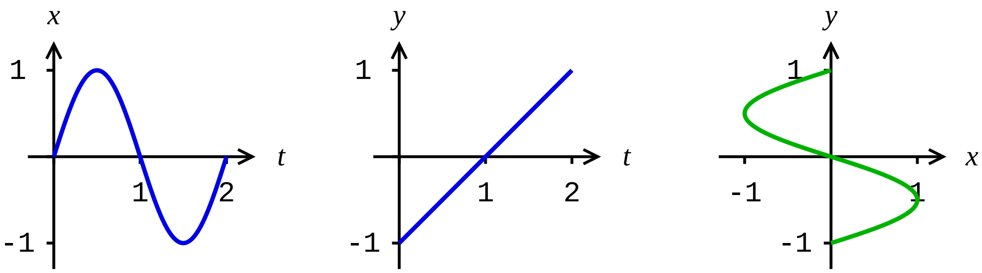
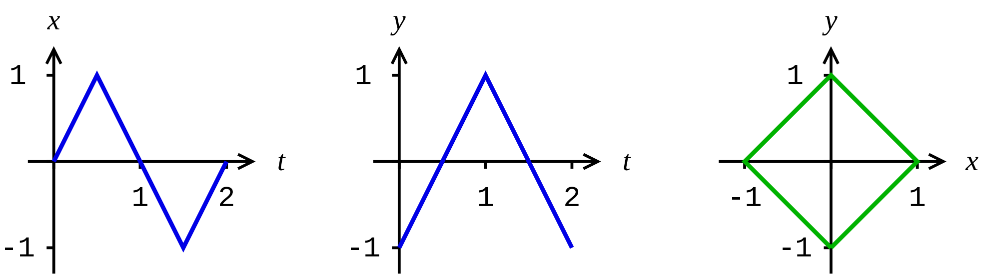
a.



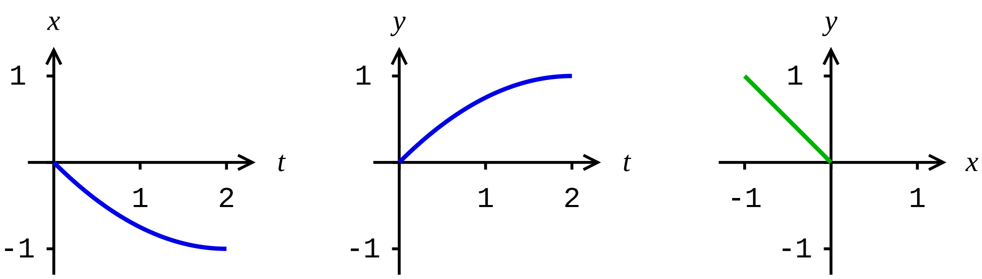
b.



c.



d.



10.2: Calculus on Parametrized Curves

Calculating $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ *along a parametrized curve*

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}}$$

Replace y with $\frac{dy}{dx}$ to find the second derivative of y with respect to x:

$$\frac{d}{dx}\boxed{} = \frac{\dfrac{d}{dt}\boxed{}}{\dfrac{dx}{dt}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

10.2.re1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ along the given curve. Then find the equation of the line tangent to the curve at the given time t.

a. $x = 4\sin t$, $y = 2\cos t$, $t = \pi/4$

b. $x = e^t$, $y = te^{2t}$, $t = 0$

c. $x = 2t^2 + 3$, $y = t^4$, $t = -1$

d. $x = t - \sin t$, $y = 1 - \cos t$, $t = \pi/3$

When $\dfrac{dx}{dt} \neq 0 = \dfrac{dy}{dt}$, the curve has a **horizontal** tangent line.

When $\dfrac{dx}{dt} = 0 \neq \dfrac{dy}{dt}$, the curve has a **vertical** tangent line.

10.2.re2. Find the times when the given curve's tangent line is horizontal or vertical.

a. $x = t^2 - t$, $y = t^3 - t$

b. $x = te^t$, $y = e^{t^2}$

c. $x = \frac{t^2}{t-1}$, $y = \frac{t}{t^2+1}$

d. $x = t - 2\sin t$, $y = 1 - \cos t$ $(0 \leq t \leq 2\pi)$

Arclength of a parametrized curve

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

10.2.re3. Find the length of the given curve.

a. $x = \sqrt{3}t^2 - 1$, $y = t - t^3$, $0 \le t \le 1$
b. $x = \cos t$, $y = t + \sin t$, $0 \le t \le \pi$
c. $x = \frac{1}{2}t^2$, $y = \frac{1}{3}(2t+1)^{3/2}$, $0 \le t \le 4$
d. $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$ $(0 \le t \le \pi/4)$

10.2.re4. Find the area swept out by this curve when it is rotated about the given axis.

a. $x = \sqrt{3}t^2 - 1$, $y = t - t^3$, $0 \le t \le 1$, y-axis
b. $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$, x-axis

Answers

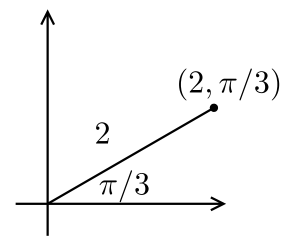
10.2.re1a. $\frac{dy}{dx} = -\frac{1}{2}\tan t$. $\frac{d^2y}{dx^2} = -\frac{1}{8}\sec^3 t$. $y = -\frac{1}{2}x + 2\sqrt{2}$ 10.2.re1b. $\frac{dy}{dx} = (2t+1)e^t$. $\frac{d^2y}{dx^2} = 2t+3$. $y = x-1$ 10.2.re1c. $\frac{dy}{dx} = t^2$. $\frac{d^2y}{dx^2} = 1/2$. $y = x-4$ 10.2.re1d. $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. $\frac{d^2y}{dx^2} = \frac{\cos t - \cos^2 t + \sin^2 t}{(1-\cos t)^3}$. $y = \sqrt{3}x + 2 - \pi/\sqrt{3}$ 10.2.re2a. hor@ $t = \pm 1/\sqrt{3}$. vert@ $t = 1/2$. 10.2.re2b. hor@$t = 0$. vert@ $t = 1$ 10.2.re2c. hor@ $t = -1$. vert@ $t = 0, 2$. 10.2.re2d. $\frac{dy}{dx} = \frac{\sin t}{1-2\cos t}$. hor@$t = 0, \pi, 2\pi$. vert@$t = \pi/3, 5\pi/3$. 10.2.re3a. 2 10.2.re3b. 4 10.2.re3c. 12. 10.2.re3d. $\frac{1}{2}\ln 2$. 10.2.re4a. $2\pi((3\sqrt{3})/5 + 1/\sqrt{3} - 2)$
10.2.re4b. $8\pi^2$

10.3: Polar coordinates

A coordinate system is a way of giving directions, in form of a pair of numbers, to a point in the plane. For instance, the rectangular coordinates (x, y) direct us to the point x units to the right (left, if $x < 0$) and y units up (down, if $y < 0$) from the origin.

The **polar coordinate** system gives direction to a point using two numbers r and θ. To reach the point, stand at the origin, facing the direction of the positive x-axis. Turn θ radians counterclockwise (clockwise, if $\theta < 0$), then move r units forward (backward, if $r < 0$). (Counterclockwise and clockwise are more simply referred to as the the **positive** and **negative** directions, respectively.)

For instance, the polar coordinates $(2, \pi/3)$ direct us to the point in the first quadrant two units from the origin along the ray that is $\pi/3$ radians from the x-axis:



In general,

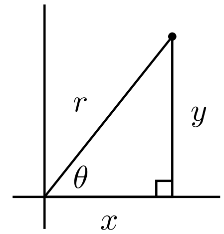
$$(r, \theta + 2n\pi), \quad (-r, \theta + (2n+1)\pi)$$

all represent the same point in the plane for any integer n. If $r \neq 0$, no other polar coordinates describe the same point as these. (If $r = 0$, then the point is the origin regardless of θ.)

Polar vs. Rectangular coordinates

Practice writing these four relationships between (r, θ) and (x, y) with the help of this picture (drawn as if r were positive and θ were acute):

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$



10.3.re1. Covert the equation from one coordinate system to the other and identify its graph.

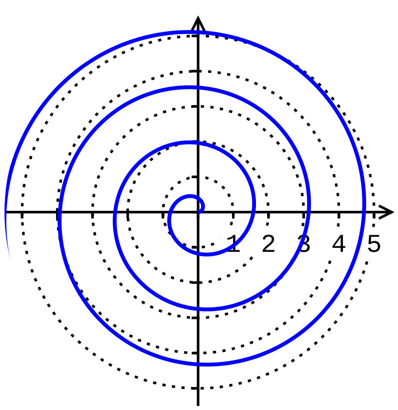
a. $y = 4$	b. $y = x$	c. $r\cos\theta = -1$
d. $r^2 = 9$	e. $r^2\sin(2\theta) = 2$	f. $r = 3\sec\theta$

Graphs of polar equations

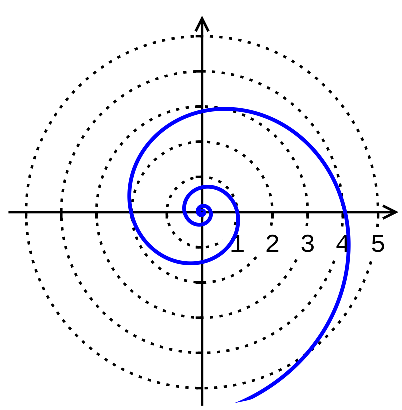
See http://kunklet.people.cofc.edu/MATH220/220notes.pdf, starting on p. 117, for some general advice and examples of graphing in polar coordinates. You should be comfortable graphing the following polar curves:

Spirals	$r > 0$ an increasing or decreasing function
Circles	$r =$ constant, $r = a\sin\theta$, $r = a\cos\theta$
Roses	$r = a\cos(n\theta)$ or $r = a\sin(n\theta)$
Limaçons	$r = b + a\cos\theta$ or $r = b + a\sin\theta$

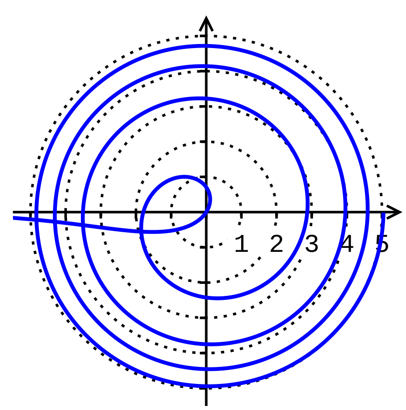
10.3.re2. Some spirals:



$r = \frac{1}{4}\theta$



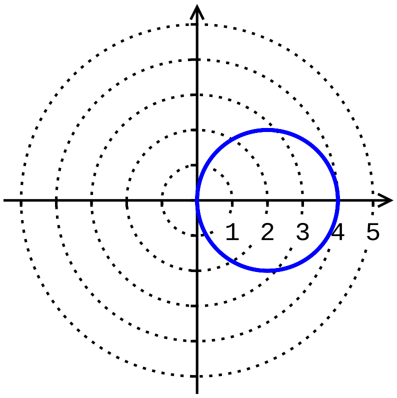
$r = (0.8)^\theta$



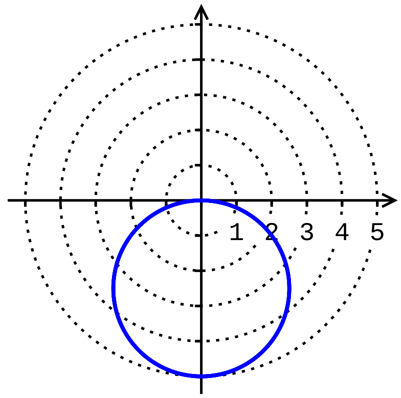
$r = \ln\theta \ (\theta > 0)$

Whenever r is getting closer to zero, the graph of $r = r(\theta)$ is spiraling in towards the origin, and whenever r is getting farther from zero, $r = r(\theta)$ is spiraling out away from the origin.

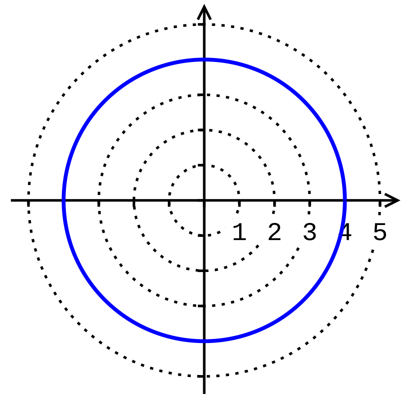
10.3.re3. Some circles:



$r = 4\cos\theta$

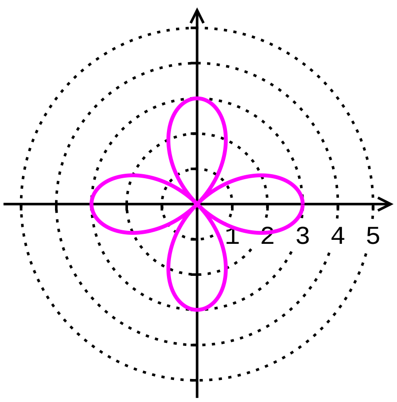


$r = -5\sin\theta$

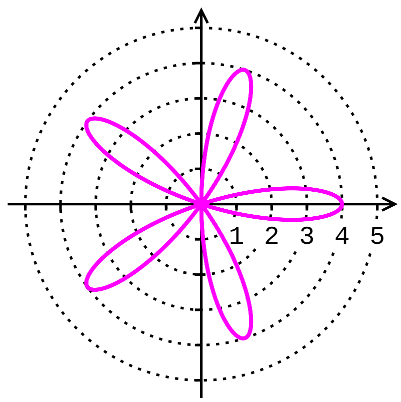


$r = 4$

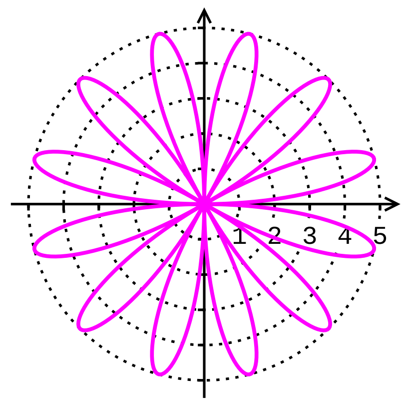
10.3.re4. Some roses:



$r = 3\cos 2\theta$



$r = 4\cos 5\theta$



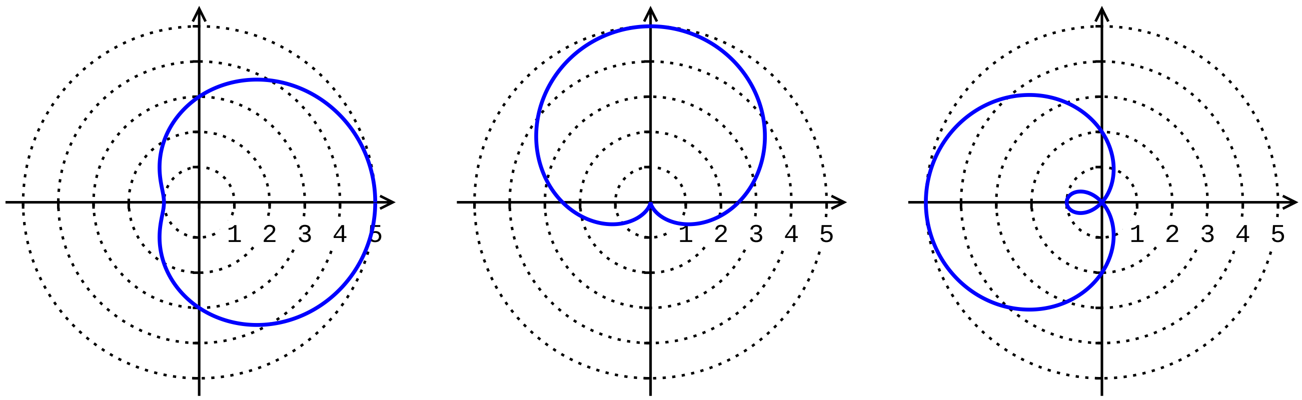
$r = 5\sin 6\theta$

Roses are composed of petals. The length of each petal from the origin to its farthest point is $|a|$, and the number of petals on the rose is

$$\begin{cases} n & \text{if } n \text{ is odd, and} \\ 2n & \text{if } n \text{ is even.} \end{cases}$$

To graph more roses, go to
https://www.desmos.com/calculator/zxgo9rhhzv

10.3.re5. Some limaçons:



$r = 3 + 2\cos\theta$ $\quad$ $r = 2.5 + 2.5\sin\theta$ $\quad$ $r = 2 - 3\cos\theta$

$1 \le r \le 5$ $\quad$ $0 \le r \le 5$ $\quad$ $-1 \le r \le 5$

A limaçon's shape depends on the $a \pm b$, maximum and minimum values of r. The graph touches the origin if and only if $r = 0$ for some θ and contains an inner loop if and only if r changes sign (that is, if its maximum $> 0 >$ its minimum.)
For an interactive graph of limaçons, visit

https://www.desmos.com/calculator/yr4rfqf5eh

10.3.re6. Graph the polar equation by hand. Check your answers on Desmos.com or another grapher. Polar graph paper is available at
http://kunklet.people.cofc.edu/MATH220/polargraph6.pdf

a. $r^2 = 16$ b. $r = -3$ c. $r = 2\sin\theta$ d. $r = 3\sin 2\theta$
e. $r = 4\cos 3\theta$ f. $r = 5\sin 4\theta$ g. $r = |\theta|$ h. $r = e^\theta$
i. $r^2 = \cos 2\theta$ j. $r = 2 + 3\sin\theta$ k. $r = 2 - 3\sin\theta$ l. $r = 3 - 2\cos\theta$
m. $r = 3.5 + 1.5\sin\theta$ n. $r = 2 + 2\sin\theta$

Slope along a polar curve

We don't need a special formula to find $\dfrac{dy}{dx}$ along a polar curve of the form $r = r(\theta)$, because such a curve is given parametrically by

$$x = r(\theta)\cos\theta \qquad y = r(\theta)\sin\theta,$$

and therefore

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

10.3.re7. Find $\dfrac{dy}{dx}$ along the polar curve.

a. $r = 3 + \sin\theta$ b. $r = \sin\theta + \cos\theta$ c. $r = 3/(\sin\theta + \cos\theta)$
d. $r = \cos^2\theta$ e. $r = \cos\theta$ f. $r = e^\theta$

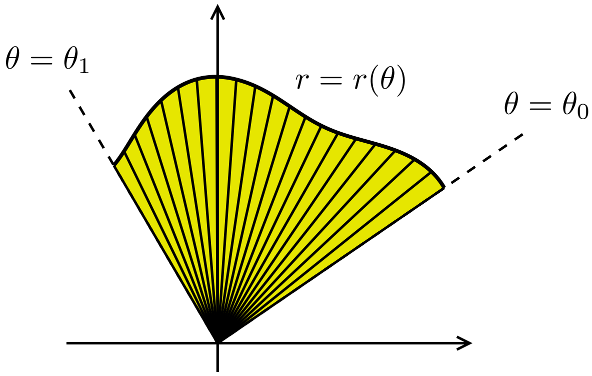
10.3.re8. Find the values of θ at which the tangent line to $r = e^{\theta}$ is horizontal or vertical.

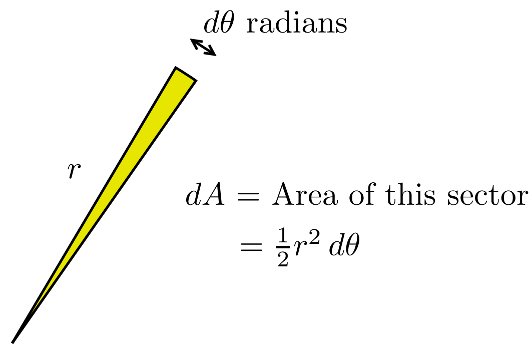
Answers

10.3.re1a. $4 = r\sin\theta$ (horizontal line) 10.3.re1b. $\theta = \pi/4$ (line through (0,0) with slope 1) 10.3.re1c. $x = -1$ (vertical line) 10.3.re1d. $x^2 + y^2 = 9$ (circle, radius 3, center (0,0)) 10.3.re1e. $y = 1/x$ (hyperbola) 10.3.re1f. $x = 3$ (vert. line) 10.3.re7a. $\frac{dy}{dx} = (\cos\theta(3 + 2\sin\theta)/(\cos^2\theta - (3 + \sin\theta)\sin\theta)$ 10.3.re7b. $\frac{dy}{dx} = (\cos(2\theta) + \sin(2\theta))/\cos(2\theta - \sin 2\theta)$ 10.3.re7c. $\frac{dy}{dx} = -1$ 10.3.re7d. $\frac{dy}{dx} = \frac{1}{3}(2\tan\theta - \cot\theta)$ 10.3.re7e. $\frac{dy}{dx} = (-\cos^2\theta + \sin^2\theta)/2\cos\theta\sin\theta$ 10.3.re7f. $\frac{dy}{dx} = (\cos\theta + \sin\theta)/(\cos\theta - \sin\theta)$, or $(1 + \tan\theta)/(1 - \tan\theta)$ 10.3.re8. Horizontal when $\cos\theta = -\sin\theta$, or $\theta = -\pi/4 + n\pi$. Vertical when $\cos\theta = \sin\theta$, or $\theta = \pi/4 + n\pi$

10.4: Area and arclength in polar coordinates

Area inside a polar curve





Area problems in polar coordinates often require the use of the half-angle formulas

$$\cos^2 x = \tfrac{1}{2}\big(1 + \cos(2x)\big) \qquad \sin^2 x = \tfrac{1}{2}\big(1 - \cos(2x)\big),$$

which can obtained by adding and subtracting

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ 1 &= \cos^2 x + \sin^2 x \end{aligned}$$

10.4.re1. Find the inside the polar curve.

a. $r = 4 + 2\cos\theta$

b. one leaf of $r = \sin(2\theta)$

c. one loop of $r^2 = 4\cos(2\theta)$

d. inner loop of $r = 1 - 2\sin(\theta)$

10.4.re2. Find the area inside both both curves.

a. $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

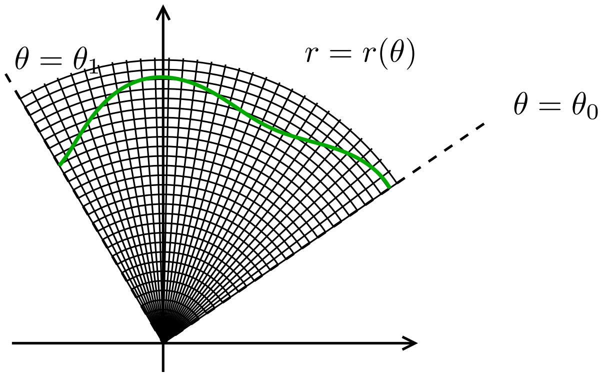
b. $r = 1$ and $r = \sqrt{2}\sin\theta$

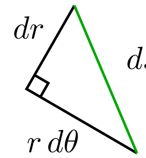
10.4.re3. Find the inside the first curve and outside the second.

a. $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

b. $r = 1 + \cos\theta$ and $r = \cos\theta$

Arclength of a polar curve





$$ds = \sqrt{(dr)^2 + (r\,d\theta)^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$$

10.4.re4. Find the length of the polar curve.

a. $r = \sin^2(\theta/2)$

b. $r = \sqrt{1 + \sin(2\theta)}$

Answers

10.4.re1a. 18π 10.4.re1b. $\pi/8$ 10.4.re1c. 2 10.4.re1d. $2(\sqrt{3} - 2) + (\pi - \sqrt{3})/2$
10.4.re2a. $-4 + 3\pi/2$ 10.4.re2b. $\frac{1}{2}(\pi - 1)$ 10.4.re3a. 8 10.4.re3b. $5\pi/4$ 10.4.re4a. 4
10.4.re4b. $2\pi\sqrt{2}$