3.11: The Hyperbolic Trig Functions

Definition 3.11.1. *The hyperbolic sine and cosine of x are*

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

The other four hyperbolic trig functions are defined in terms of these:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

Similarities of the hyperbolic trig functions to the circular trig functions

1. $\cosh 0 = 1$ $\qquad$ $\sinh 0 = 0$

2. $\cosh x$ is an even function. $\qquad$ $\sinh x$ is an odd function.

3. $\cosh^2 x - \sinh^2 x = 1$

4. $\cosh x \cosh y + \sinh x \sinh y = \cosh(x + y)$

   $\sinh x \cosh y + \cosh x \sinh y = \sinh(x + y)$

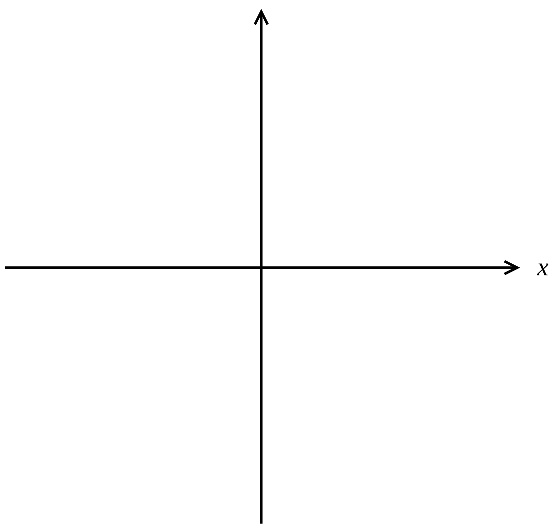
5.

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

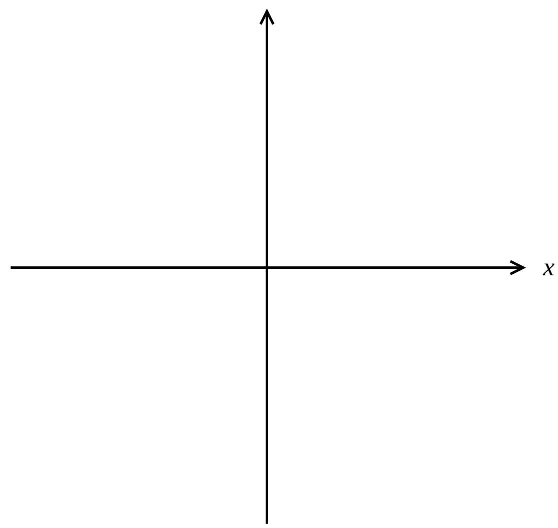
$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Graphs of the hyperbolic trig functions

y = cosh x



y = sinh x



Some limits involving the hyperbolic trig functions

3.11.e1. Evaluate the limits, if they exist.

a. $\lim_{x\to\infty} \cosh x$ b. $\lim_{x\to-\infty} \cosh x$ c. $\lim_{x\to\infty} \sinh x$

d. $\lim_{x\to-\infty} \sinh x$ e. $\lim_{x\to\infty} \tanh x$ f. $\lim_{x\to\infty} \dfrac{5e^x - e^{-2x}}{3e^x + 4e^{-2x}}$

Tip: It is far easier to remember the *way* to do problems in this chapter than the *formula*.

6.1: Areas Between Curves

6.1.e1. Find the area of the region bounded by the curves $y = x^2$ and $y = 2x$.

6.1.e1, continued. Here's another solution, obtained with a different way of slicing.

6.1.e2. Find the area of the region bounded by $y = x^2$ and $y = (x-2)^2$ and $y = 0$. What will be the easiest way to slice up this region?

6.2: Volumes

Calculating volumes by slicing

6.2.e1. A (three-dimensional) solid has for its base the region in the xy-plane bounded by $y = x^2$ and $y = 2x$. Find the volume of the solid if its cross-sections perpendicular to the x-axis are squares with one side in the plane.

Tip: To answer the question perfectly doesn't require a great drawing. Usually, it only needs to represent where one curve lies relative to the other(s).

6.2.e2. The intersection of a solid with the xy-plane is the region bounded by $y = x^2$ and $y = 2x$. Find the volume of the solid if its cross-sections perpendicular to the y-axis are circles with diameters in the plane.

See more of this type of volume problem at
http://kunklet.people.cofc.edu/MATH220/stew0602prob.pdf

Volumes of Revolution: the method of discs and washers

6.2.e3. The region bounded by $y = x^2$ and $y = 2x - x^2$ is rotated about the line $y = 0$. Find the volume swept out by the region.

6.2.e4. The region bounded by $y = x^2$ and $y = 4$ is rotated about the line $y = 4$. Find the volume swept out by the region.

6.2.e5. The region bounded by $y = x^2$ and $y = 2x$ is rotated about the line $x = 2$. Find the volume swept out by the region.

We'll see another technique for solving 6.2.e5 in the next section.

6.3: More Volumes

Volumes of Revolution: the method of cylindrical shells

Let's apply this method to the last example in the previous section:

6.2.e5, continued. The region bounded by $y = x^2$ and $y = 2x$ is rotated about the line $x = 2$. Find the volume swept out by the region. Express your answer as a definite integral.

Tip: Don't ask "Should I use shells or washers?" Instead, ask "Is it easier to slice the region vertically or horizontally?"

6.3.e1. Let R be the region in the plane bounded by $y = x$ and $y = 2 - x^2$. Find the volume swept out as R is rotated about the line $x = 4$. Express your answer as a definite integral.

6.3.e1, continued. Find the volume swept out as R is rotated about $y = 2$. Express your answer as a definite integral.

6.3.e2. Let R be the triangle with vertices $(-1, 2)$, $(0, 1)$, and $(3, 2)$.

a. Find the volume swept out by R when it is rotated about $y = 2$.

b. Find the volume swept out by R when it is rotated about $x = 3$.

c. Find the volume of the solid whose intersection with the plane is R and whose cross-sections perpendicular to the y-axis are squares with a side in the plane.

Express your answers as definite integrals.

6.4: Work

When a constant force is applied to an object moving in a straight line in the same direction as the force, the **work** done by that force is, by definition, its magnitude times the distance traveled by the object.
Work can be measured in foot-pounds or Newton-meters (also known as Joules). For instance, if I lift a 2-pound calculus book 3 feet in the air, I've done 6 foot-pounds of work.

In this section, we'll use integrals to find work in cases when the force is not constant.

6.4.e1. A rope 50 feet long hangs from the top of a tall tower. If the rope weighs 30 lbs, find the work to pull 40 ft of the rope to the top of the tower.

6.4.e2. Suppose a 10-lb weight is attached to the end of the rope in 6.4.e1. Find the work to pull the rope and weight to the top of the tower.

6.4.e3. A one-pound bucket initially holds 50 pounds of sand, but as it is lifted, the sand leaks from the bucket at a rate of $\frac{2}{3}$ pounds per foot. Find the work to lift the leaky bucket and its contents 75 feet.

Spring problems

A spring has a natural resting length. To *hold* the spring at a length different from its natural length requires a force which has the following simple description.

Hooke's Law. *The force $f(x)$ necessary to hold a spring x meters from its natural length is proportional to x. That is, for some* **spring constant** k,

$$f(x) = kx$$

To *stretch* or *compress* a spring beyond its natural length requires work.

Tips: (1) In a spring-work problem, don't confuse *force*, measured in pounds or Newtons, and *work*, measured in foot-pounds or Joules.
(2) Look for the information that will allow you to solve for the spring constant k.

6.4.e4. A 2-Newton force will hold a spring 0.1m from its resting length.

a. Find the work done in stretching the spring from 0m to 0.1m beyond its natural length.

b. Find the work needed to stretch the spring from 0.1m to 0.2m beyond its natural length.

6.4.e5. The work to pull a spring from resting length to 10cm beyond that length is 2J.

a. Find the spring constant for this spring.

b. Find the work to pull the spring from 20cm to 30cm beyond its natural length.

c*. Starting from its natural length, how far will 1J of work stretch the spring?

Tip: All work problems look alike in the following sense: If an object moves along an axis from $x = a$ to $x = b$, and if the force applied to the object when at position x is $f(x)$, then

$$dw = f(x)\,dx \qquad \text{and} \qquad w = \int_a^b f(x)\,dx$$

6.5: Average Value

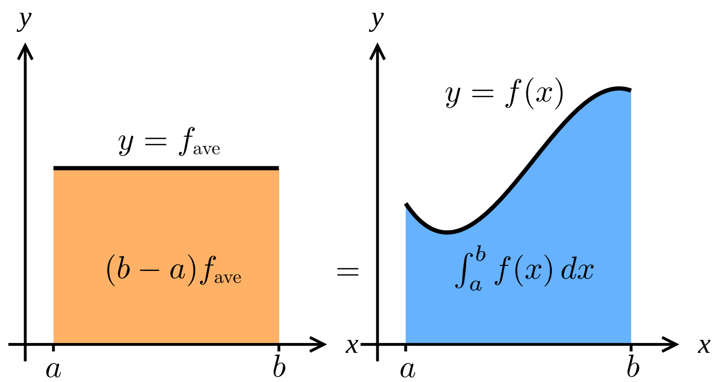
Definition 6.5.1. *The* **average value** *of the function $f(x)$ on the interval $[a, b]$ is*

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

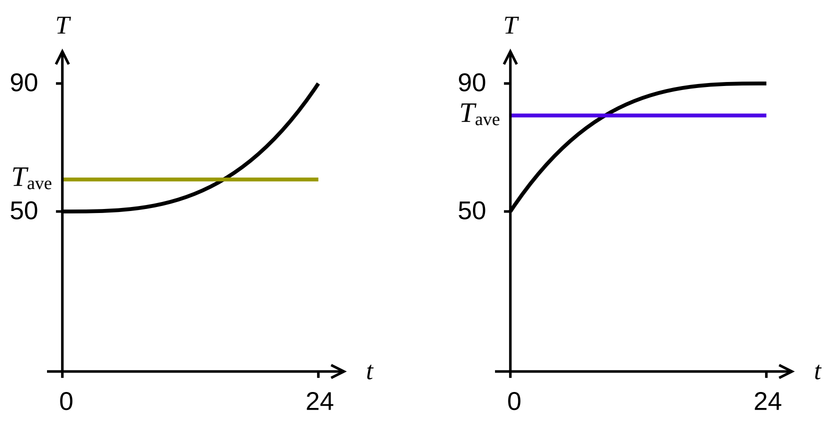
Since

$$(b-a)f_{\text{ave}} = \int_a^b f(x)\,dx,$$

f_{ave} is the altitude of the horizontal line that captures the same area as f over $[a, b]$:

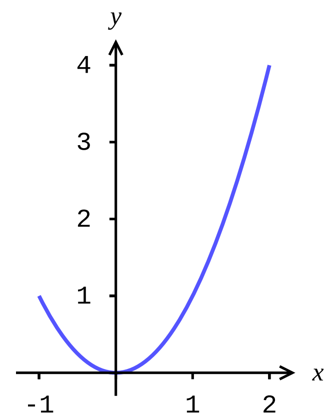


6.5.e1. The graphs below show the temperature T in downtown Charleston as a function of time t over two different days. On both days, the high was 90° and the low was 50°, but the second day had a higher average temperature. Can you see why?

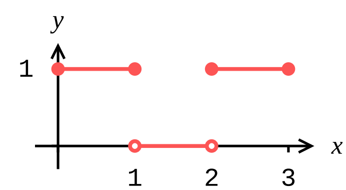


Note that the average value of a given function depends on the interval. In either of the graphs above, the average temperature over the first 12 hours of the day is lower than the average over the last 12 hours, in the same way that the average of a your first 4 quizzes might be different than the average of your last four quizzes.

6.5.e2. Calculate the average of x^2 on the interval $[-1, 2]$



6.5.e3. Calculate the average of the function graphed at right on the interval $[0, 3]$.



Mean Value Theorem for Integrals 6.5.2. *If $f(x)$ is continuous on the interval $[a, b]$, then there exists at least one number c in $[a, b]$ for which $f(c) = f_{\text{ave}}$.*

6.5.e3, continued. Find the number(s) c in $[-1, 2]$ guaranteed by the MVTI 6.5.2 for the function x^2.

One last note. When we calculate the average of an object's velocity $s'(t)$ over an interval of time $[a, b]$,

$$\frac{1}{b-a}\int_a^b s'(t)\,dt = \frac{s(b)-s(a)}{b-a}$$

we obtain the object's change in position over the change in time. That is, "average velocity" as defined in this section is the same as "average velocity" as defined in precalculus.

7.1: Integration by Parts

If u and v are differentiable functions of x, then the derivative of their product is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

and so

$$\int \left(u\frac{dv}{dx} + v\frac{du}{dx} \right) dx = uv.$$

Distribute the dx, write $\frac{du}{dx}dx = du$ and $\frac{dv}{dx}dx = dv$, and break the integral in two:

$$\int u\,dv + \int v\,du = uv.$$

Finally, subtract the second integral from both sides and we obtain the formula known as **Integration by Parts**:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

IBP is sometimes useful in integrating a product:

7.1.e1. Integrate: $\int xe^x\,dx$

Sometimes we use IBP twice in the same problem:

7.1.e2. Integrate: $\int x^2 \cos x\,dx$

There's a product in *every* integral:

7.1.e3. Integrate: $\int \ln x \, dx$

Tip: When confronted with a new problem, think of past examples that are similar, and what worked in those problems.

7.1.e4. Integrate: $\int x^3 \ln x \, dx$

Trapping the integral in an equation and solving.

7.1.e5. Integrate: $\int e^x \sin x \, dx$

In another, we derive a **Reduction Formula** that will be useful in the next section.

7.1.e6. Derive the formula $\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

For a longer list of such formulas, see the next lecture.

Odds and Ends

Substitution is a powerful tool from Calc I and we'll use frequently in Calc II. Review it and come me if you need help.

7.1.e7. Integrate: $\int xe^{x^2}\,dx$

7.1.e8. Integrate: $\int \sin\sqrt{x}\,dx$

Sometimes it's faster to Guess-and-Check than to carry out a substitution:

7.1.e9. $\int \sin(4x)\,dx$

7.2: Trigonometric Integrals

PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

Integrals involving sine and cosine

7.2.e1. $\int \sin^6 x \cos^3 x \, dx$

7.2.e2. $\int \sin^5 x \cos^7 x \, dx$

Rule 1: To integrate

$$\int \sin^n x \cos^m x \, dx,$$

if the exponent of $\sin x$ is odd, you can substitute $u = \cos x$;
if the exponent of $\cos x$ is odd, you can substitute $u = \sin x$.

Integrals involving tangent and secant

7.2.e3. $\int \tan x \, dx$

7.2.e4. $\int \sec x \, dx$

7.2.e5. $\int \tan^4 x \sec^4 x \, dx$

7.2.e6. $\int \tan^5 x \sec^5 x \, dx$

Rule 2: To integrate

$$\int \tan^n x \sec^m x \, dx,$$

if the exponent of $\sec x$ is even, you can substitute $u = \tan x$;
if the exponent of $\tan x$ is odd, you can substitute $u = \sec x$.

Reduction formulas

These can be derived using Integration by Parts, as in Example 7.1.e6.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

7.2.e7. $\int \sin^4 x\,dx$

7.2.e8. $\int \sec^3 x\,dx$

7.2.e9. $\int \tan^2 x \sec^3 x\,dx$

For more practice using these reductions formulas, see
http://kunklet.people.cofc.edu/MATH220/stew0702prob.pdf

Rule 3:
To integrate even powers of tangent and/or odd powers of secant, rewrite the integrand entirely in terms of secant and use the reduction formula.

To integrate even powers of sine and/or cosine, either

a. Rewrite entirely in terms of either sine or cosine and use a Reduction formulas,
b. use Euler's Formula (as in the next lecture), or
c. use the half-angle identities as in our text (but I don't recommend it).

Finally, sometimes integration is made easier by using one of these familiar trig formulas:

SUM & DIFFERENCE IDENTITIES

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

DOUBLE ANGLE FORMULAS

$$\sin(2x) = 2 \sin x \cos x$$
$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

HALF ANGLE FORMULAS

$$\cos^2 x = \tfrac{1}{2}\big(1 + \cos(2x)\big)$$
$$\sin^2 x = \tfrac{1}{2}\big(1 - \cos(2x)\big)$$

7.2.5: Trigonometric Integrals using Euler's Formula

This topic does not appear in our text, so I've prepared complete notes for this section, which you can find at http://kunklet.people.cofc.edu/MATH220/euler.pdf

A **complex number** is an expression of the form $x + iy$, where x and y are real numbers and i is the imaginary square root of -1. The set of all complex numbers is denoted $\mathbb{C}$.

The numbers x and y are called **real part** and **imaginary part** of $x + iy$, denoted

$$x = \mathrm{Re}(x + iy) \qquad y = \mathrm{Im}(x + iy).$$

The **conjugate** of $x + iy$, denoted $\overline{x + iy}$, is defined as

$$\overline{x + iy} = x - iy.$$

We can add, subtract, multiply, and divide complex numbers using the familiar associative and distributive laws of arithmetic, remembering that $i^2 = -1$.

7.2.5.e1. Compute the following.

a. $(2 + 3i) + (4 - 9i)$

b. $(4 - i)(1 - 3i)$

7.2.5.e2. Use the conjugate to divide in $\mathbb{C}$:

$$\frac{4 - i}{1 + 3i} = \frac{(4 - i)}{(1 + 3i)} \frac{(1 - 3i)}{(1 - 3i)} = \frac{1 - 13i}{1^2 - (3i)^2} = \frac{1}{10} - \frac{13}{10} i$$

Most of the elementary functions on the real line can be extended in a natural way to complex the complex plane. Polynomials and rational functions, for instance can be evaluated at complex numbers using only complex arithmetic.

7.2.5.e3. If $p(z) = z^2 - 3z + 4$, then

$$\begin{aligned} p(2 + 3i) &= (2 + 3i)^2 - 3(2 + 3i) + 4 \\ &= 4 + 12i + 9i^2 - 6 - 9i + 4 \\ &= -7 + 3i. \end{aligned}$$

The exponential function can be evaluated at complex numbers using **Euler's Formula**:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

(so that $e^{x+iy} = e^x \cos y + ie^x \sin y$). Since $\cos\theta$ is an even function and $\sin\theta$ is odd,

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

By adding and subtracting these two equations we obtain formulas for the sine and cosine in terms of complex exponentials:

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Euler's formula is proven using **power series** that we'll study in later in this course, but we can use it now for some trig integrals.

7.2.5.e4. To integrate $\int \sin(3x)\cos(2x)\,dx$, first rewrite the integrand using Euler's formula:

A **Sinusoidal** function is one of the form

$$A\sin(Bx + C) + D \quad \text{or} \quad A\cos(Bx + C) + D$$

Any function of the form

$$\sin(mx)\sin(nx) \quad \text{or} \quad \sin(mx)\cos(nx) \quad \text{or} \quad \cos(mx)\cos(nx)$$

can be written as a sum of sinusoidal functions as in 7.2.5.e4 (and then easily integrated).

7.2.5.e5. Integrate $\int \sin^2(3x)\,dx$ by first writing the integrand as a sum of sinusoidals.

In problems like 7.2.5.e5, it pays to be able to quickly and accurately expand powers of $\frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\frac{e^{i\theta}-e^{-i\theta}}{2i}$. The formula we used in 7.2.5.e5,

$$(x+y)^2 = x^2 + 2xy + y^2$$

is a special instance of the **Binomial Theorem**, which tells us how to expand binomials of the form $(x+y)^n$ using **Pascal's Triangle**:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

See http://kunklet.people.cofc.edu/MATH220/euler.pdf for details and exercises.

7.2.5.e6. Generate the next two rows of Pascal's Triangle.

7.2.5.e7. Expand the following.

a. $(x+y)^4$ b. $(x+y)^6$ c. $(x-y)^6$

7.2.5.e8. Write as a sum of sinusoidal functions.

a. $\cos^4 x$

b. $\sin^6 x$

Summary of 7.2 Trig integrals and 7.2.5 Euler's formula:

1. $\int \sin(Ax)\sin(Bx)\,dx$, $\int \sin(Ax)\cos(Bx)\,dx$, $\int \cos(Ax)\cos(Bx)\,dx$: Use Euler.
2. $\int \sin^m x \cos^n x\,dx$
   a. If either m or n is odd, substitution as in 7.2 is easiest.
   b. If m and n are even, either use
      i. Euler
      ii. A reduction formula, if you have it.
3. $\int \tan^m x \sec^n x\,dx$
   a. If m is odd or n is even, substitution as in 7.2 is easiest.
   b. If m is even and n is odd, rewrite in terms of secant and use reduction formula, as in 7.2.e9.

7.3: Trigonometric Substitution

This technique of integration is useful when the integrand includes a quadratic polynomial under a fractional or negative exponent and polynomial substitution is not possible, e.g.,

$$\int x^2(x^2-1)^{3/2}\,dx \quad \text{and} \quad \int \frac{dx}{(x^2+2x+2)^2}$$

7.3.e1. Use a polynomial substitution to rewrite the integral:

a. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+1}}$ b. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+1}}$

Quadratics of the form Ax^2+B (A and $B>0$)

Rewrite the quadratic using $\tan^2\theta+1=\sec^2\theta$.

7.3.e2. $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$

7.3.e3. $\displaystyle\int \sqrt{x^2+4}\,dx$

Quadratics of the form $B - Ax^2$ (A and $B > 0$)

Rewrite the quadratic using $1 - \sin^2\theta = \cos^2\theta$.

7.3.e4. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

7.3.e5. $\displaystyle\int \frac{x^2}{\sqrt{4-9x^2}}\,dx$

Quadratics of the form $Ax^2 - B$ (A and $B > 0$)

Rewrite the quadratic using $\sec^2\theta - 1 = \tan^2\theta$.

7.3.e6. $\displaystyle\int \frac{1}{(x^2-4)^{3/2}}\,dx$

Completing the square

Completion of the square—followed by a change of variable—allows us to rewrite every quadratic polynomial as one without a x^1 term:

7.3.e7. $\displaystyle\int \frac{1}{(x^2-4x)^{3/2}}\,dx$

7.3.e8. $\displaystyle\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$

Summary of 7.3 Trig Substitution:

Turn this quadratic	into this trig expression;	you may use
$Ax^2 + B$	$\tan^2\theta + 1$	$\sqrt{\sec^2\theta} = \sec\theta$
$B - Ax^2$	$1 - \sin^2\theta$	$\sqrt{\cos^2\theta} = \cos\theta$
$Ax^2 - B$	$\sec^2\theta - 1$	$\sqrt{\tan^2\theta} = \tan\theta$

When your quadratic is of the form $ax^2 + bx + c$, complete the square and make a linear change of variable to eliminate the bx term.

7.4: Partial Fractions

Partial fractions is a technique that allows us to rewrite a rational function as a sum of simpler rational functions, provided we can factor the denominator.

7.4.e1. $\displaystyle\int \frac{5x-1}{x^2-x-2}\,dx$

To **equate coefficients** means to set equal the corresponding coefficients of two equal polynomials.

To rewrite a rational function as we did in 7.4.e1 requires that the degree of the numerator be *less than* the degree of the denominator. Fortunately, **polynomial long division** ensures that that's never a problem.

Fact 7.4.1. *If $p(x)$ and $d(x)$ are polynomials, then there exist polynomials $q(x)$ and $r(x)$ for which*

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \text{ and } \deg r(x) < \deg d(x).$$

7.4.e2. $\displaystyle\int \frac{x^3+2x^2-6x-3}{x^2-9}\,dx$

The Partial Fraction Decomposition

Recall that a quadratic polynomial $ax^2 + bx + c$ with real coefficients is **irreducible** if it can't be factored over the real numbers. A quadratic is irreducible if and only if any of the following are true.

i. $ax^2 + bx + c$ can't be factored over the real numbers.
ii. $ax^2 + bx + c$ has no linear factors with real coefficients.
iii. $ax^2 + bx + c$ has no real zeros.
iv. $ax^2 + bx + c = a(x - h)^2 + k$ where $ak > 0$.
v. $b^2 - 4ac < 0$

The next fact says that linear and irreducible quadratic polynomials play the same roll in the polynomials as prime numbers play in the integers.

Fact 7.4.2. *Every polynomial with real coefficients factors uniquely (up to nonzero constant factors) into powers of linear and irreducible quadratic factors.*

The rational function $\dfrac{p(x)}{d(x)}$ is called **proper** if $\deg p < \deg d$.

7.4.e3. $\dfrac{5x-1}{x^2-x}$ is proper; $\dfrac{2x^2-6x}{x^2-9}$ and $\dfrac{x^3}{x^2-6x+9}$ are not.

Fact 7.4.3. *Every proper rational function* $\dfrac{p(x)}{d(x)}$ *can be written uniquely as a sum of rational functions called its* **Partial Fraction Decompsition (PFD)** *with the following rules. (Below, A, B, C, … are constants.)*

a. For each linear factor $(ax + b)^k$ of $d(x)$, the PFD includes

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{D}{(ax+b)^k},$$

b. For each irreducible quadratic factor $(ax^2 + bx + c)^\ell$ of $d(x)$, the PFD includes

$$\frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \cdots + \frac{Ex+F}{(ax^2+bx+c)^\ell}.$$

7.4.e4. Find the *form* of the PFD of $\dfrac{p(x)}{(x-1)(x^2+2x-3)^2(x^2+2x+3)}$, assuming it is proper.

Equating coefficients is sometimes necessary, but often there are easier ways to find some of the constants in a PFD.

7.4.e5. Find the PDF. Precede with long division if necessary.

a. $\dfrac{2x^2+5x+5}{(x+2)^2(x+1)}$ b. $\dfrac{x^4+x^2+x-18}{x^3-x^2+4x-4}$

There are more partial fraction practice problems at
http://kunklet.people.cofc.edu/MATH220/stew0704prob.pdf.

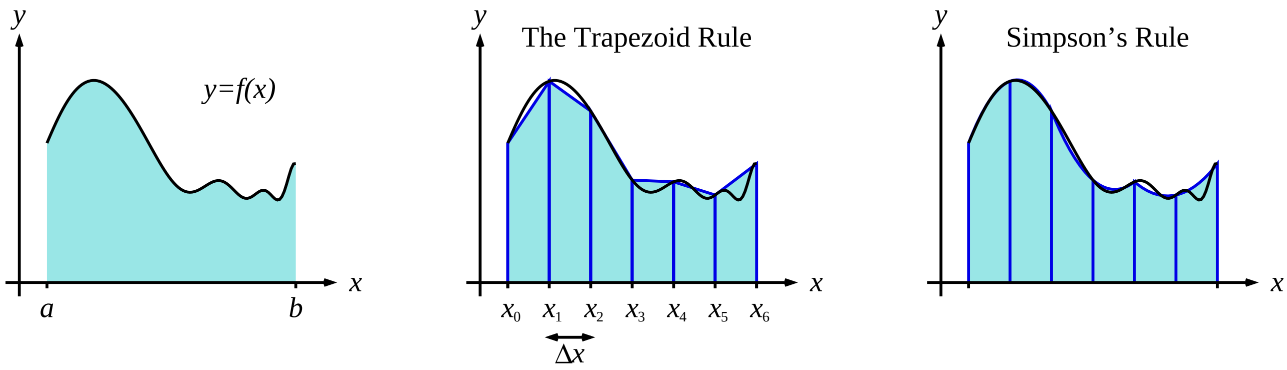
7.4.e6. Integrate.

a. $\displaystyle\int \left(\frac{1}{2x-1} - \frac{3}{(2x-1)^2} \right) dx$ b. $\displaystyle\int \frac{x-5}{x^2+4}\, dx$ c. $\displaystyle\int \frac{dx}{x^2+2x+2}$

7.4.e7. What trig substitution should you make in $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}$?

7.7: Numerical Integration

Numerical integration, or quadrature, is the numerical approximation of a definite integral that we're unable to evaluate directly. In this course, we'll discuss two examples of **quadrature rules**.



The **Trapezoid Rule** and **Simpson's Rule** for n subintervals are

$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\big)$$
$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)$$

Simpson's Rule requires n to be even.
The **Error** of either of these is defined to be the integral minus the approximation .

$$E_T = \int_a^b f(x)\,dx - T_n \qquad\qquad E_S = \int_a^b f(x)\,dx - S_n$$

7.7.e1. Approximate $\int_0^\pi \sin x\,dx$ using the trapezoid rule and Simpson's rule with $n = 6$.

I calculated these and found

$$T_6 = 1.9541 \qquad S_6 = 2.00086 \qquad E_T = 0.0459 \qquad E_S = -0.00086$$

7.7.e2. Approximate $\int_0^\pi \sin(x^2)\,dx$ using the trapezoid rule and Simpson's rule with $n = 6$.

$$T_6 =$$
$$\frac{\pi}{12}\big(\sin 0 + 2\sin(\frac{\pi^2}{36}) + 2\sin(\frac{4\pi^2}{36}) + 2\sin(\frac{9\pi^2}{36}) + 2\sin(\frac{16\pi^2}{36}) + 2\sin(\frac{25\pi^2}{36}) + \sin(\pi^2)\big)$$

and

$$S_6 =$$
$$\frac{\pi}{18}\big(\sin 0 + 4\sin(\frac{\pi^2}{36}) + 2\sin(\frac{4\pi^2}{36}) + 4\sin(\frac{9\pi^2}{36}) + 2\sin(\frac{16\pi^2}{36}) + 4\sin(\frac{25\pi^2}{36}) + \sin(\pi^2)\big)$$

I calculated these and found $T_6 = 0.608622$ and $S_6 = 0.906737$.

You can see some slides of the trapezoid and Simpson's rule at work on this problem at http://kunklet.people.cofc.edu/MATH220/707figb.pdf

It's impossible to know the error of our approximation unless we also already know the true value of the integral. Fortunately, the size of the derivatives of the integrand determine upper bounds for the absolute error in the Trapezoid and Simpson's. (The key to the following fact is Taylor's Theorem, which we'll see when we discuss power series later in the course.)

Fact 7.7.1. *If K and L are numbers so that*

$$|f''(x)| \le K \qquad \text{and} \qquad |f^{(4)}(x)| \le L$$

for all x in the interval $[a,b]$, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4}$$

In particular, under these hypotheses, $|E_T|$ and $|E_S|$ both go to zero as $n \to \infty$.

7.7.e1, continued. $|f''(x)| = |f^{(4)}(x)| = |\sin x| \le 1$, so we can take $K = L = 1$. Combining 7.7.1 when $n = 6$ with the errors we observed directly in 7.7.e1,

$$0.0459 = |E_T| \le \frac{1(\pi-0)^3}{12\cdot 6^2} = 0.0717$$
$$0.00086 = |E_S| \le \frac{1(\pi-0)^5}{180\cdot 6^4} = 0.00131$$

If we can't maximize $|f''(x)|$ or $|f^{(4)}(x)|$, it may still be easy to find a legitimate K and L, with the help of these two properties of absolute values:

$$|AB| = |A||B| \qquad \text{and} \qquad |A+B| \le |A| + |B|$$

7.7.e2, continued. Using $|x| \le \pi$ and $|\sin x^2|$ and $|\cos x^2| \le 1$, we find

$$|f''(x)| = |2\cos(x^2) + 4x^2\sin(x^2)|$$

so 7.7.1 promises that

$$|E_T| \le \frac{42\pi^2}{12 \cdot 6^2} = 0.9595.$$

Similarly,

$$\begin{aligned} |f^{(4)}(x)| &= |-48x^2\cos x^2 - 12\sin x^2 + 16x^4\sin x^2| \\ &\le |-48x^2\cos x^2| + |-12\sin x^2| + |16x^4\sin x^2| \\ &= 48x^2|\cos x^2| + 12|\sin x^2| + 16x^4|\sin x^2| \\ &\le 48\pi^2 \cdot 1 + 12 \cdot 1 + 16\pi^4 \cdot 1 \\ &\le 2045, \end{aligned}$$

so

$$|E_S| \le \frac{2045\pi^5}{180 \cdot 6^4} = 2.6826.$$

Needless to say, an error potentially as large as either of these is unacceptable. Increasing n, e.g., to 32, reduces the worst-possible errors to

$$|E_T| \le \frac{42\pi^3}{12 \cdot 32^2} = 0.1060 \quad \text{and} \quad |E_S| \le \frac{2045\pi^5}{180 \cdot 32^4} = 0.0033.$$

A typical problem in approximation is to find the necessary n to achieve some prescribed error bound, as in the next example.

7.7.e3. How large must n be in order to approximate $\int_0^\pi \sin(x^2)\,dx$ with an absolute error less that 10^{-8} using

a. the Trapezoid rule. b. Simpson's Rule.

(ans: a. $n > 104,173.88$, so and b. $n > 767.87$, so)

For more error analysis problems, see
http://kunklet.people.cofc.edu/MATH220/stew0707prob.pdf

4.4: l'Hospital's Rule and indeterminant forms.

When taking limits,

$$\frac{\text{nonzero}}{0} \implies \lim = \pm\infty \qquad\qquad \frac{\text{finite}}{\infty} \implies \lim = 0.$$

But

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}$$

are called **indeterminate forms** because they tell us nothing about the actual value of the limit. Fortunately, there's a tool for these situations.

l'Hôpital's Rule 4.4.1. *If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \textit{ or } \pm\infty$$

and if

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \textit{ exists and also equals } L.$$

Notes on l'Hospital's Rule

1. $x \to a$ can be replaced throughout by $x \to \infty$, $x \to -\infty$, $x \to a^-$, or $x \to a^+$.
2. L can be ∞, $-\infty$, or any real number.
3. l'Hôpital's Rule n̂ever allows us to conclude that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.

Indeterminate forms and examples of each

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

l'Hospital's Rule applies only to the first two of these. Any others would have to be rewritten as a quotient in order to use l'Hôpital's Rule.

4.4.e1. $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos x}$

4.4.e2. $\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x}$

4.4.e3. $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x}$

4.4.e4. $\displaystyle\lim_{x\to 0^+} x^p \ln x \qquad (p>0)$

4.4.e5. $\displaystyle\lim_{x\to\infty} \big(\ln(3x-1)-\ln(2x+1)\big)$

When the variable appears in both the exponent and the base, take the limit of the logarithm:

4.4.e6. $\lim_{x \to 0^+} x^x$

4.4.e7. $\lim_{x \to \infty} x^{\frac{k}{\ln x}}$

4.4.e8. $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x$

7.8: Improper Integrals

A definite integral $\int_a^b f(x)\,dx$ is said to be **improper** if $f(x)$ has a vertical asymptote somewhere on $[a,b]$, or if a or b is infinite.

7.8.e1. Each of the following integrals is improper.

a. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$ b. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$ c. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x+2)^2}$

The Fundamental Theorem of Calculus cannot be applied to an improper integral. Instead, we use this rule:

> An improper integral is defined to be the limit of proper integrals.

An improper integral is said to **converge** or **diverge**, depending on whether this limit exists or not.

For lack of a better term, we'll any x-values where the integrand has a vertical asymptote, or any infinite endpoints of the interval **bad points**.

7.8.e1, continued. Find the bad points of each of the integrals above.

Rules for evaluating an improper integral

> 1. Rewrite the integral as a sum, if necessary, so that the bad points are endpoints.
> 2. Allow only one bad point per integral.
> 3. When the improper integral has been written as a sum, require each integral in the sum to converge in order for the sum to converge.

7.8.e1, continued. Evaluate the improper integral, if it converges.

a. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x}}\,dx$ b. $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$ c. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x+2)^2}$

The p-integral

The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ is called a p**-integral**.

7.8.e2. Determine how convergence or divergence of $\int_1^\infty \frac{1}{x^p}\,dx$ depends on p.

Conclusion:

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ converges if } p > 1, \text{ and diverges to } \infty \text{ if } p \leq 1.}$$

For instance $\int_1^\infty \frac{1}{x^{1.001}}\,dx$ converges, but $\int_1^\infty \frac{1}{x^{0.999}}\,dx$ diverges to infinity.

Fact 7.8.1. *If $f(x) \geq 0$ and $\int_a^b f(x)\,dx$ is improper, then either $\int_a^b f(x)\,dx = \infty$ or $\int_a^b f(x)\,dx$ converges (to a finite number).*

7.8.e3. Depending on p, $\int_1^\infty x^{-p}\,dx$ either converges to a finite number or diverges to ∞. In contrast, $\int_0^\infty \sin x\,dx$ does neither.

Sometimes, when improper integral is impossible to evaluate, we can still determine whether or not it converges:

Comparison Test 7.8.2. *If $f(x) \geq g(x) \geq 0$ on $[a, b]$, and if $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are both improper, then*

if $\int_a^b f(x)\,dx$ converges, then so must $\int_a^b g(x)\,dx$.

Equivalently,

if $\int_a^b g(x)\,dx$ diverges to infinity, then so must $\int_a^b f(x)\,dx$.

7.8.e4. Determine the convergence or divergence of the improper integral.

a. $$\int_2^\infty \frac{dx}{\sqrt{\sqrt{x}-1}}$$

b. $$\int_3^\infty \frac{dx}{x^4+4}$$

Challenge Problem: a. Integrate $\int \frac{dx}{\sqrt{\sqrt{x}-1}}$ by making a substitution.

b. Integrate $\int \frac{dx}{x^4+4}$ by first factoring the demoninator. Hint: find a and b so that $x^4+4 = (x^2+ax+2)(x^2+bx+2)$.

More examples

7.8.e5. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$

7.8.e6. $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$

7.8.e7. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2-x}\,dx$

8.1: Arclength

Arclength means the length of a curve. To find it, we imagine chopping up the curve into infinitely many pieces, each infinitesimally small. Let ds to be the length of the piece of the curve at the point (x, y):

$$\boxed{s = \int ds \qquad \text{where} \quad ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Let's test drive this on an example where we already know the answer:

8.1.e1. Find the length of $y = 5x$ on $2 \leq x \leq 4$.

Arclength problems are sometimes carefully prepared to make the integration easy. The trick is to recognize a perfect square under the radical when it's there.

8.1.e2. Find the length of $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ on $1 \leq x \leq 2$.

8.1.e3. Find the length of the curve $y = \ln(x^2 - 1)$ on $2 \leq x \leq 3$.

8.2: Surface areas of revolution

A picture will help:

$$dA = \begin{cases} 2\pi x\, ds & \text{when rotating about the } y\text{-axis} \\ 2\pi y\, ds & \text{when rotating about the } x\text{-axis} \end{cases}$$

Write the integral in terms of a single variable in order to integrate.

8.2.e1. Find the area swept out by the curve $y = x^2$ for $0 \leq x \leq 3$ as it is rotated about the y-axis.

8.2.e2. Find the area swept out by $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ for $1 \leq x \leq 2$ as it is rotated about

a. the y-axis

b. the x-axis.

8.2.e3. Find the area of the a sphere of radius r by viewing it as a surface of revolution.

11.1: Sequences and their limits

A **sequence** is a function defined on the integers, usually the nonnegative integers $n \geq 0$ or the positive integers $n > 0$.

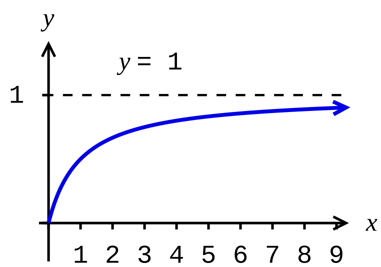
Even though a sequence is a function, it has its own notation different from the way we usually write functions in calculus.

$$\begin{aligned} u, v, w, x, y, z &= \text{typical names of complex- or real-valued variables} \\ i, j, k, l, n, m &= \text{typical names of integer-valued variables (except when } i = \sqrt{-1}\text{)} \\ f(x) &= \text{the value of the function } f \text{ at the real number } x \\ a_n &= \text{the value of the sequence } \{a_n\} \text{ at the integer } n \end{aligned}$$

Examples & Useful Facts

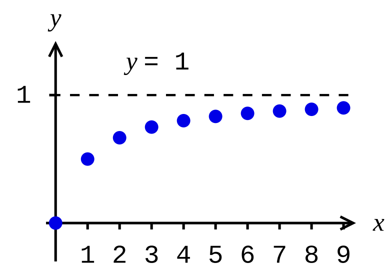
11.1.e1.

The graph of $y = f(x) = \dfrac{x}{x+1}, \quad x \geq 0$



$$\lim_{x\to\infty} \frac{x}{x+1} = 1$$

The graph of $y = a_n = \dfrac{n}{n+1}, \quad n \geq 0$



$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

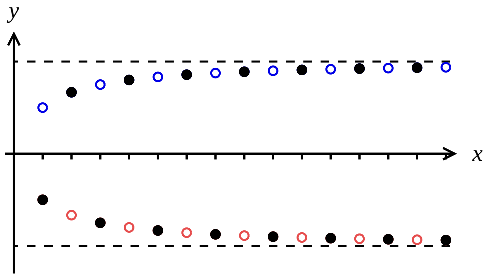
Useful fact 11.1.1. *If $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.*

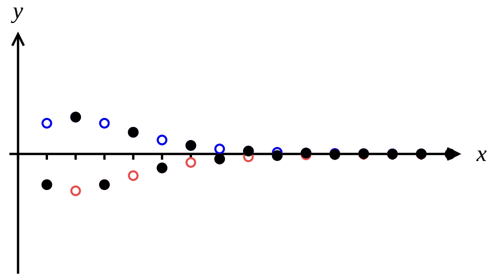
11.1.e2. Multiple ways of writing the same sequence:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\} \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad \left\{\frac{n}{n+1}\right\} \qquad a_n = \frac{n}{n+1}$$

11.1.e3. Find the limit of the given sequence.

a. $\left\{\dfrac{(-1)^n n}{n+1}\right\}$ b. $\left\{\dfrac{(-1)^n n}{n^2+1}\right\}$





Useful fact 11.1.2. $\lim_{n\to\infty} a_n = 0$ *if and only if* $\lim_{n\to\infty} |a_n| = 0$.

As 11.1.e3.a demonstrates, 11.1.2 is not true if 0 is replaced by any other number.

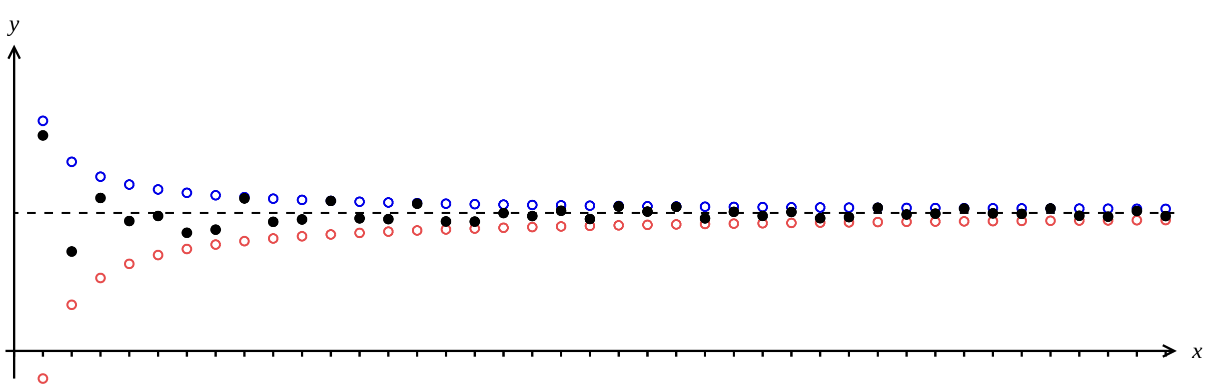
Useful fact 11.1.3. *If* $a_n \leq b_n$ *for all* n*, and if* $\lim_{n\to\infty} a_n = \infty$*, then* $\lim_{n\to\infty} b_n = \infty$.

11.1.e4. $n!$ (read "n **factorial**") is defined as

$$n! = \begin{cases} n\cdot(n-1)\cdots 2\cdot 1 & \text{if } n > 0, \text{ and} \\ 1 & \text{if } n = 0. \end{cases}$$

The limit $\lim_{n\to\infty} n! = \infty$, since $n! \geq n$ for all n.

Useful fact 11.1.4. *If* $a_n \leq b_n \leq c_n$ *for all* n*, and if* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$*, then* $\lim_{n\to\infty} b_n$ *exists and also equals* L.



Famous Limits Everybody Should Know (FLESK)

1. If r is a constant, then $\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1, \\ 1 & \text{if } r = 1, \text{ and} \\ 0 & \text{if } -1 < r < 1, \end{cases}$ but does not exist if $r \le -1$.

2. If c is a positive constant, then $\lim_{n\to\infty} c^{1/n} = 1$.

3. If p is a real constant, then $\lim_{x\to\infty} x^p = \begin{cases} \infty & \text{if } p > 0, \\ 1 & \text{if } p = 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$

4. If k is a real constant, then $\lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^x = e^k$.

5. If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\text{lead term of } p(x)}{\text{lead term of } q(x)}$$

Consequently, $$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} (\text{lead term of } p(x))$$

and $$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} \pm\infty & \text{if } \deg p > \deg q, \\ \dfrac{\text{lead coefficient of } p}{\text{lead coefficient of } q} & \text{if } \deg p = \deg q, \text{ and} \\ 0 & \text{if } \deg p < \deg q. \end{cases}$$

6. If $p(x)$ is a polynomial, then $\lim_{x\to\infty} \frac{p(x+1)}{p(x)} = 1$.

7. If $p(x)$ is a nonzero polynomial, then $\lim_{x\to\infty} |p(x)|^{1/x} = 1$.

We saw FLESK 4 earlier in 4.4.e8. Next, we'll look at some example of these limits. You can read more about these in http://kunklet.people.cofc.edu/MATH220/flesk.pdf

11.1.e5. Find the limit of the following sequences

a. $\{3^n\}$ b. $\{(-\frac{4}{5})^n\}$ c. $\{(-\frac{5}{4})^n\}$

11.1.e6. Find the limit of the following sequences

a. $\{(1024)^{1/n}\}$ b. $\{n^{0.01}\}$ c. $\{n^{-0.01}\}$

11.1.e7. Find the limits.

a. $\lim_{n\to\infty} \left(\frac{n+3}{n}\right)^n$

b. $\lim_{n\to\infty} |1+n-4n^3|^{1/n}$

c. $\lim_{x\to\infty} \frac{4x^3+2x-1}{-3x^4-x^2+9x}$

d. $\lim_{x\to\infty} \frac{4x^4-2x-1}{-3x^4-x^2+9x}$

e. $\lim_{x\to\infty} \frac{4x^5-2x-1}{-3x^4-x^2+9x}$

f. $\lim_{x\to\infty} \frac{4(x+1)^3+2(x+1)-1}{4x^3+2x-1}$

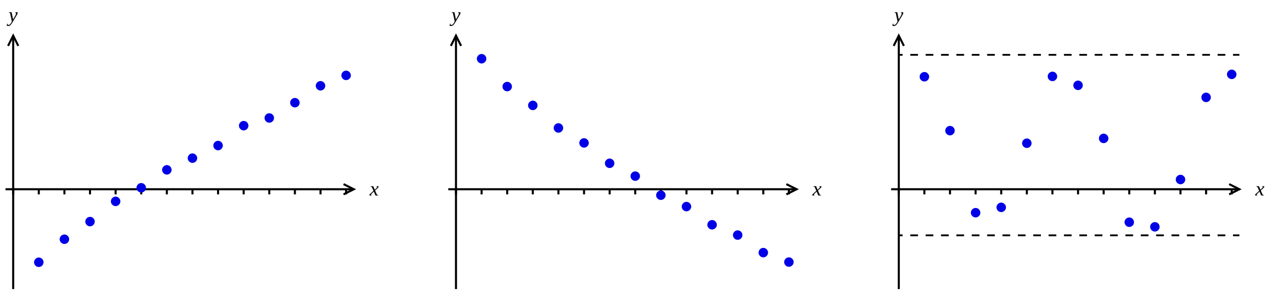
FLESK 1 does not apply to 11.1.e7 a; FLESK 2 does not apply to 11.1.e7 b.

Monotonicity & Boundedness

A sequence $\{a_n\}$ is ...	if, for some B and for all n, ...
increasing	$a_n < a_{n+1}$
decreasing	$a_n > a_{n+1}$
monotonic	the sequence $\{a_n\}$ is increasing or decreasing
bounded above	$a_n < B$
bounded below	$a_n > B$
bounded	the sequence $\{a_n\}$ is bounded above and below

11.1.e8. This is what its graph would look like if a sequence were ...

increasing **decreasing** **bounded**



Facts 11.1.5.

1. *Every increasing sequence is* ________________________________.
2. *Every decreasing sequence is* ________________________________.
3. *An increasing sequence is convergent if and only if it is* ____________________.
4. *A decreasing sequence is convergent if and only if it is* ____________________.

It's impossible to tell that a sequence is increasing or decreasing from its first terms. The surest way to test for these is to take a derivative.

11.1.e9. Determine the monotonicity and boundedness of the sequence ($n \geq 1$).

a. $\{\ln n\}$ b. $\{e^{-n}\}$ c. $\{\sin(n^2)\}$ d. $\left\{\dfrac{n}{(n+4)^2}\right\}$

11.2: Series and their sums

Sigma ($\sum$) notation

If a_i is a sequence, then $\sum_{i=1}^{n} a_i$ stands for the the sum $a_1 + a_2 + \cdots + a_n$.

11.2.e1. $\sum_{i=1}^{4} \frac{1}{i^2} =$

$\{a_1, a_2, a_3, \ldots\}$ is an **infinite sequence**. Its **limit** denoted $\lim_{i\to\infty} a_i$.

$a_1 + a_2 + a_3 + \cdots$ is an **infinite series**. Its **sum** denoted $\sum_{i=1}^{\infty} a_i$.

$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the nth **partial sum** of the series.

Definition 11.2.1. $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$, *if this limit exists. That is, the sum of a series is defined to be the limit of its partial sums.*

Note the similarity of infinite series to improper integrals.
It is important to note that the symbols $\lim\limits_{n\to\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ **do not** mean the same thing:

11.2.e2. Determine the convergence or divergence of

a. the sequence $\{1, 1, 1, 1, 1, \ldots\}$

b. the series $1 + 1 + 1 + 1 + \cdots$

Decimal expansions are infinite series:

11.2.e3. Write $1.2122122212222122222 1\cdots$ as an infinite series. Why must it converge?

Thee's a simple way to express a repeating decimal as a rational number. See https://web.ma.utexas.edu/users/mks/302/repdec.html.

Geometric and Telescoping series

Definition 11.2.2. *A* **geometric series** *is one of the form*

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{j=0}^{\infty} ar^j$$

for some numbers a and r.

As seen in class,

Fact 11.2.3. *If $a \neq 0$, then $\sum_{j=0}^{\infty} ar^j = \frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.*

11.2.e4. Find the sum of the series, if it converges.

a. $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \cdots$

b. $\sum_{n=2}^{\infty} \frac{3^{2n-2}}{5^{n+1}}$

Definition 11.2.4. *A* **telescoping series** *is one of the form* $\sum_{k=0}^{\infty}(b_{k+1} - b_k)$ *for some sequence* b_k*.*

This type of series is named for the collapsing telescope, like the one seen here, because of the way their partial sums "collapse" to a few terms.

11.2.e5. Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$.

The nth term test

Most series other than geometric or telescoping series are difficult to sum, and in the next sections we'll learn several tests that may help us conclude that a series converges or diverges. The author of our textbook called the first of these the Test for Divergence, but in many other texts it goes by a more descriptive name:

The nth term test 11.2.5. *If the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges, then* $\displaystyle\lim_{n\to\infty} a_n$ *must equal 0.*

Since the convergence of $\lim_{n\to\infty} a_n$ is usually easier to determine than the convergence of $\sum_{n=1}^{\infty} a_n$, we most often use the nth term test in this form:

$$\textit{If } \lim_{n\to\infty} a_n \textit{ fails to equal zero, then the series } \sum_{n=1}^{\infty} a_n \textit{ must diverge.}$$

11.2.e6. What does the nth term test say about the following series?

a. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

b. $\sum_{n=1}^{\infty} \ln\left(\frac{n+2}{n}\right)$

Example 11.2.e6b. demonstrates that the converge of the nth term test is false. If the nth term of a series goes to zero, it doesn't necessarily follow that the series must converge. That is, the nth term test *sometimes* allows us to conclude that a series diverges. It *never* allows us to conclude that a series converges.

11.2.e7. As we'll see in the next section, the **Harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though its nth term $\frac{1}{n} \to 0$ as $n \to \infty$.

Useful rules 11.2.6. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *both converge and if c is any number, then*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \text{ and}$$
$$\sum_{n=1}^{\infty} c \cdot a_n = c\left(\sum_{n=1}^{\infty} a_n\right).$$

However,

$$\sum_{n=1}^{\infty}(a_n b_n) \neq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

11.3 and 11.4: nonnegative series

The tests we learn in section 11.3 and 11.4 apply only to **nonnegative series**, that is, series with no negative terms, and they reply on the following fundamental fact.

Fact 11.3.1. *If $a_n \geq 0$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ either diverges to ∞ or converges (to a finite sum).*

11.3: The Integral Test

The Integral Test 11.3.2. *If $f(x)$ is positive and decreasing on the interval $[k, \infty)$ for some number k, then*

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

11.3.e1. The **p-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. (See 7.8.e2.)

If $p \leq 0$:

If $p > 0$:

11.3.e2. Determine the convergence/divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+9}$

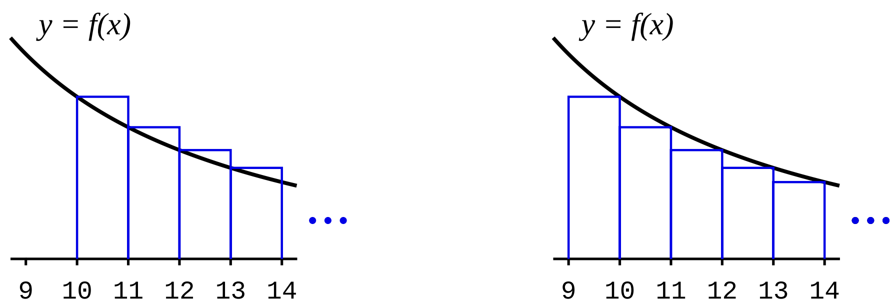
Error analysis in the integral test

We can see why the integral test is true in three steps, and along the way, learn a useful error bound.

1. $\int_1^\infty f(x)\,dx$ and $\int_{k+1}^\infty f(x)\,dx$ must both converge or both diverge, since the difference between them is the finite number $\int_1^{k+1} f(x)\,dx$.

2. Likewise, $\sum_{n=1}^\infty f(n)$ and $\sum_{n=k+1}^\infty f(n)$ must both converge or both diverge, since the difference between them is the finite number $\sum_{n=1}^k f(n)$.

3. Suppose for the moment that $k = 9$:



More generally, even if $k \neq 9$,

$$\boxed{\int_{k+1}^\infty f(x)\,dx \;\le\; \sum_{n=k+1}^\infty f(n) \;\le\; \int_k^\infty f(x)\,dx}$$

Note that this provides upper and lower bounds for the **error**

$$\sum_{n=1}^\infty f(n) - \sum_{n=1}^k f(n) = \sum_{n=k+1}^\infty f(n)$$

when we use the kth partial sum to approximate the sum of the series.

11.3.e3. Bound the error that occurs when $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is approximated by its 10th partial sum $\sum_{n=1}^{10} \frac{1}{n^4}$.

11.3.e3, continued. If I wish for the error to be less than 10^{-8}, how many terms should I use in my approximation?

For more practice problems like11.3.e3, see
http://kunklet.people.cofc.edu/MATH220/stew1103prob.pdf.

11.4: The Comparison Test

As in 11.3, in this section we limit our attention to nonnegative series. These have the special property that they can only converge (to a finite sum) or diverge to infinity.

The Comparison Test 11.4.1. *Suppose* $0 \leq a_n \leq b_n$, *for all* $n \geq k$ *for some* k.

If $\sum_{n=1}^{\infty} b_n$ *converges,* $\sum_{n=1}^{\infty} a_n$ *must also converge.*

If $\sum_{n=1}^{\infty} a_n$ *diverges,* $\sum_{n=1}^{\infty} b_n$ *must also diverge.*

11.4.e1. Determine the convergence/divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+\pi)}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ c. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{\sqrt{n^3-n^2}}$

11.4.e2. Determine the convergence/divergence of the series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4 + 1}}$

It sometimes happens that our series and the series we would compare to it aren't related in a way that leads to any conclusion via the Comparison Test. In that event, there's another test we might try.

Limit Comparison Test 11.4.2. *If a_n and b_n are both ≥ 0 for all n, and if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive (and finite) number, then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

11.4.e2, continued.

Even if $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ is 0 or ∞, all is not lost.

Limit Comparison Test, continued.

If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, *then* $0 \le a_n < b_n$ *for all* $n > k$ *for some number* k.

If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, *then* $a_n > b_n \ge 0$ *for all* $n > k$ *for some number* k.

In either of these events, we can then try the ordinary comparison test.

11.4.e3. Determine the convergence/divergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

11.5: The Alternating Series Test

An **alternating series** is one that alternates in sign, such as

$$-\frac{1}{2}+\frac{2}{3}-\frac{1}{4}+\frac{4}{9}-\frac{1}{8}+\frac{8}{27}-\cdots \tag{11.5.1}$$

Our next test sometimes allows us to conclude that an alternating series converges.

Alternating Series Test 11.5.2. *If $b_n \geq b_{n+1}$ for all n in $[k,\infty)$ for some number k, and if $\lim_{n\to\infty} b_n = 0$, then the alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ converges.*

Note that $$\sum_{n=0}^{\infty}(-1)^n b_n = b_0 - b_1 + b_2 - b_3 + \cdots$$
is -1 times $$\sum_{n=0}^{\infty}(-1)^{n+1} b_n = -b_0 + b_1 - b_2 + b_3 - \cdots,$$
so if either converges, so does the other.

11.5.e1. The **Alternating Harmonic Series** $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges by the Alternating Series Test.

11.5.e2. Determine the convergence/divergence of the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$.

When the Alternating Series Test fails to apply

11.5.e3. Determine the convergence/divergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2n}$.

If $\lim_{n\to\infty} b_n$ fails to be zero, by Useful Fact 11.1.2, $\lim_{n\to\infty}(-1)^n b_n$ cannot equal zero, and therefore $\sum_{n=0}^{\infty}(-1)^n b_n$ must diverge by the nth term test.

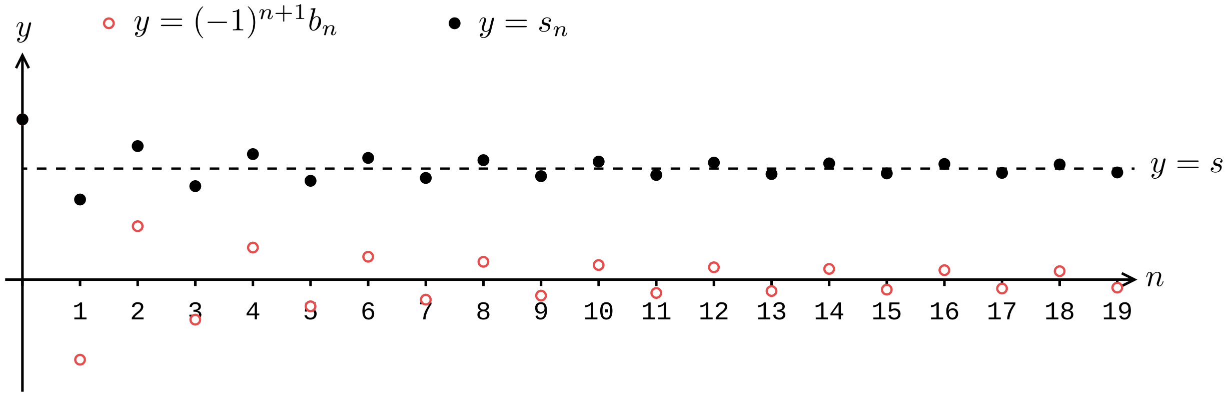
If b_n fails to be nondecreasing, then the Alternating Series Test is **inconclusive**.

(In fact, the series (11.5.1) converges, even though its sequence of absolute values fails to be nondecreasing.)

Error analysis in the alternating series test

Let's assume the hypothesis of the Alternating Series Test and ask why the conclusion must be true.

$$\begin{aligned} s &= b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + \cdots \\ s_0 &= b_0 \\ s_1 &= b_0 - b_1 \\ s_2 &= b_0 - b_1 + b_2 \\ s_3 &= b_0 - b_1 + b_2 - b_3 \\ s_4 &= b_0 - b_1 + b_2 - b_3 + b_4 \\ &\vdots \end{aligned}$$



Alternating Series Test, continued. *The sum of the series lies between any two consecutive partial sums.*

11.5.e4. *Suppose we approximation the sum s of the alternating harmonic series by the partial sum of its first 100 terms:*

$$s \approx s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{100}.$$

What can we say about magnitude and sign of the error $s - s_{100}$ of this approximation?

What we saw in 11.5.e4, is true in general: if s is the sum of the alternating series, and s_n its nth partial sum, then the **Alternating Series Test, continued**, implies the following:

1. The absolute error $|s - s_n| \leq |s_{n+1} - s_n|$, that is, the absolute value of the $(n+1)$th term of the series.

2. The sign of the error $s - s_n$ equals the sign of $s_{n+1} - s_n$, that is, the sign of the $(n+1)$th term of the series.

11.5.e4, continued. How many terms of the alternating harmonic series must I add to be certain that the partial sum is within 10^{-4} of the sum of the series?

For more practice problems like11.5.e4, see http://kunklet.people.cofc.edu/MATH220/stew1105prob.pd

11.6: Absolute Convergence and the Root and Ratio Tests

Absolute and conditional convergence

Fact 11.6.1. *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ must also converge.*

Definition 11.6.2. *We say the series $\sum_{n=1}^{\infty} a_n$ is* **absolutely convergent** *if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

11.6.e1. Show by example that $\sum_{n=1}^{\infty} |a_n|$ needn't converge if $\sum_{n=1}^{\infty} a_n$ is convergent.

Definition 11.6.3. *We say the series $\sum_{n=1}^{\infty} a_n$ is* **conditionally convergent** *if $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is divergent.*

11.6.e2. Give an example of an absolutely convergent series and a conditionally convergent series.

The Root and Ratio Tests 11.6.4. *Let L be the following limit, if it exists.*

ROOT TEST	RATIO TEST
$L = \lim_{n\to\infty} \sqrt[n]{\|a_n\|}$	$L = \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|}$

Then, regardless which of these you used to compute L,

if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and

if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges (in fact, $\lim_{n\to\infty} |a_n| = \infty$).

Note that the Root and Ratio Tests are inconclusive if $L = 1$.

11.6.e3. Apply the Root test to the geometric series $\sum_{n=0}^{\infty} ar^n$.

11.6.e4. Apply the Ratio test to the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

11.6.e5. Apply the Ratio test to the series $\sum_{n=0}^{\infty} (-1)^n 2^{n+1} \frac{1}{n!}$.

> The Root and Ratio tests will work poorly on a series that resembles a p-series but will work well on a series that resembles a geometric series or includes a factorial factor.

AC/CC/D?

11.6.e6. Determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n^2+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+2}{3^n}$ c. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{2^n+1}$

11.7: Strategy for Testing Series

See the text for a helpful list of tips and examples of how to choose a convergence test for an infinite series.

11.7.e1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{10^{2n}}{(n+1)!}$ c. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{2n+1}\right)^n$ d. $\displaystyle\sum_{n=1}^{\infty} ne^n$

e. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ f. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n - 4^n}$ g. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$ h. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^4 - 2n^3 + 2}$

11.8: Introduction to Power Series

Definition 11.8.1. *A* **power series** *in x* **centered at** *a is a series of the form*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Its **coefficients** *are the numbers $c_0, c_1, c_2, \ldots$.*

The nth coefficient c_n may depend on n but not on x.

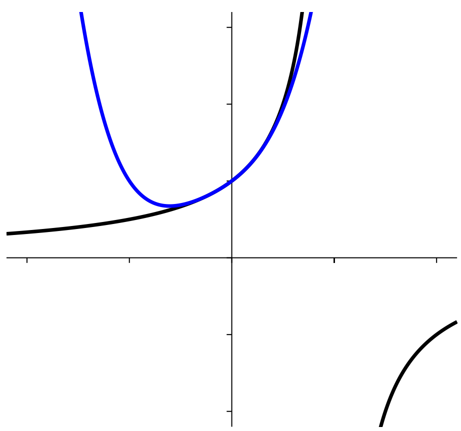
11.8.e1. This geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

is a power series centered at 0. It converges if and only if $-1 < x < 1$, and, when it does, its sum is $\frac{1}{1-x}$, meaning $\frac{1}{1-x}$ is the limit of the partial sums of the series:

$$\begin{aligned}
&1\\
&1+x\\
&1+x+x^2\\
&1+x+x^2+x^3\\
&\vdots
\end{aligned}$$

For instance, here's a graph of the fourth-degree partial sum of the series and the sum of the series:



$$y = \frac{1}{1-x} \quad \text{and} \quad y = 1 + x + x^2 + x^3 + x^4$$

You can see more of these at http://kunklet.people.cofc.edu/MATH220/geopsums.pdf

Tip: to determine where a power series converges, start with the Root or Ratio Test.

11.8.e2. Where does the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converge?

Theorem 11.8.2. *For any power series centered at $x = a$, exactly one of the following is true:*

1. *The series converges only at $x = a$.*
2. *The series converges for all real x.*
3. *There is a number $R > 0$ with the property that the series*
   converges absolutely if $|x - a| < R$, and
   diverges if $|x - a| > R$.

In case 3., the power series converges on the interval $(a - R, a + R)$, and can converge absolutely, converge conditionally, or diverge at the endpoints, where $x - a = \pm R$.
The set of points at which a power series converges is called its **Interval of Convergence**, and the half-length of that interval is called its **Radius of Convergence**. Theorem 11.8.2 tells us that the center of a power series is always the center of its interval of convergence.

11.8.e3. What is the radius of convergence in cases 1, 2, and 3 in the theorem above?

11.8.e4. Find the interval and radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n(n^2+1)}$.

11.8.e5. Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

You can see some partials sums of this series at
http://kunklet.people.cofc.edu/MATH220/taylormystery.pdf
What do you guess to be the sum of this series?

11.9: Obtaining Power Series Representations of Given Functions

We often wish to find a power series that sums to a given function. In this section we explore how to do this by manipulating another series, either by algebra or by calculus.

Obtaining one power series from another by algebraic manipulation

11.9.e1. Find a power series representation of the given function and its interval of convergence.

a. $\dfrac{1}{1+2x}$ b. $\dfrac{1}{1-4x^2}$ c. $\dfrac{x^3}{1-4x^2}$

Obtaining one power series from another by differentiation and integration

Theorem 11.9.1. *One can differentiate and integrate the sum of a power series term-by-term (provided it converges at more than just one point). Doing so produces a series with the same radius of convergence as the original.*

11.9.e2. Find a power series representation of the given function and its radius of convergence.

a. $\dfrac{1}{(1-x)^2}$ b. $\dfrac{1}{(1+x)^3}$ c. $\ln(1+x)$

It is often useful to express integrals and solutions to differential equations as power series:

11.9.e3. Find a power series representation of the indefinite integral $\int \frac{dx}{1+x^{10}}$.

11.9.e3, continued. Use your answer to express $\int_0^{1/2} \frac{dx}{1+x^{10}}$ as the sum of an alternating series.

11.10: Taylor Series

If a power series converges to a function at more than just one point, its coefficients are completely determined by the values of the function and its derivatives at the center of the series.

To see what this means, suppose that $f(x)$ is the sum of a power series centered at a with a positive radius of convergence:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \end{aligned}$$

Evaulate at $x = a$:

$$f(a) = c_0$$

Because the series has a positive radius of convergence, we can differentiate it term by term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \\ &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \end{aligned}$$

and evaulate again at $x = a$:

$$f'(a) = c_1$$

Again:

$$\begin{aligned} f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} \\ &= 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots \\ f''(a) &= 2c_2 \end{aligned}$$

Again:

$$\begin{aligned} f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-3} \\ &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + 6\cdot 5\cdot 4c_6(x-a)^3 + \cdots \\ f'''(a) &= 3\cdot 2c_3 = 3!c_3 \end{aligned}$$

The pattern continues:

$$\begin{aligned} f^{(4)}(a) &= 4\cdot 3\cdot 2c_4 = 4!c_4 \\ f^{(5)}(a) &= 5\cdot 4\cdot 3\cdot 2c_5 = 5!c_5 \\ &\vdots \\ f^{(n)}(a) &= n!c_n \end{aligned}$$

Theorem 11.10.1. *If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with some positive radius of convergence, then, for all n,*

$$c_n = \frac{1}{n!} f^{(n)}(a). \tag{11.10.2}$$

Definition 11.10.3. *If $f(x)$ possesses derivatives of all orders at $x = a$,* **Taylor series** *for $f(x)$ centered at a is*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Theorem 11.10.1 says that the Taylor series for f at a is the only power series centered at a that could possibly sum to $f(x)$.

Definition 11.10.4. *The* **Maclaurin series** *for the function $f(x)$ is the Taylor series for $f(x)$ centered at 0:*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$$

11.10.e1. Find the Maclaurin series for the given function.

a. e^x b. $\sin x$ c. $\ln(1+x)$

The following is a consequence of Taylor's Theorem, which we'll see in section 11.11.

Fact 11.10.5. *Each of the following functions is the sum of its Maclaurin series.*

$$\begin{array}{llll} \dfrac{1}{1-x} & \ln(1-x) & \tan^{-1} x & \sin^{-1} x \\ \sin x & \cos x & e^x & (1+x)^k \end{array}$$

The Maclaurin series for $\sin^{-1} x$ is derived in exercise 11.10.51. The others are collected in Table 1, p. 768. The best way to learn these series is to practice deriving them.

11.10.e2. Find the Taylor series for the given function centered at a.

a. $e^x \quad (a = -1)$ b. $x^3 - 2x^2 + 3 \quad (a = 1)$ c. $\sin(x^3) \quad (a = 0)$

Definition 11.10.6. *The* **binomial series** *is the Maclaurin series for* $(1+x)^k$. *The* **binomial coefficient** $\binom{k}{n}$ *defined to be the coefficient of* x^n *in the binomial series for* $(1+x)^k$, *allowing us to write the binomial series simply as*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

$\binom{k}{n}$ *is read "k choose n" and is given by the formula*

$$\binom{k}{n} = \begin{cases} 1 & \text{if } n = 0, \text{ and} \\ \dfrac{\overbrace{k(k-1)(k-2)\cdots}^{n \text{ factors}}}{\underbrace{n(n-1)(n-2)\cdots}_{n \text{ factors}}} & \text{if } n \neq 0 \end{cases}$$

Discovered independently by Newton and Gregory, the binomial series was the earliest power series expansion of a function other than the geometric series. Taylor derived the formula 11.10.2 from the binomial series. The simple proof seen at the beginning of this section is due to MacLauren. For more on the history of series expansions of functions, see *Mathematical Thought from Ancient to Modern Times*, by Morris Kline.

11.10.e3. Find the Maclaurin series of the given function:
a. $(1+x)^{1/2}$

b. $(1+x)^4$

Fact 11.10.7. *The radius of convergence of the binomial series* $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ *is*

$$\begin{cases} \infty & \text{if } k \text{ is a nonnegative integer, and} \\ 1 & \text{otherwise.} \end{cases}$$

11.11: Taylor Polynomials and Taylor's Theorem

Definition 11.11.1. *The nth* **Taylor polynomial** *centered at a for $f(x)$ is the partial sum of its Taylor series centered at a, up to and including the degree-n term:*

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k. \tag{11.11.2}$$

Compare with Definition 11.10.3.

11.11.e1. Find the fourth Taylor polynomial $T_4(x)$ centered at 1 for $x \ln x$.

11.11.e2. Find the third and fourth Taylor polynomials $T_3(x)$ and $T_4(x)$ centered at 0 for $\sin x$.

Perhaps you'll recognize

$$T_1(x) = f(a) + f'(a)(x-a)$$

as the linearization of $f(x)$ at $x = a$ from Calc I. The graph of T_1 is the line tangent to the graph of f at $x = a$, because $T_1(x)$ has the same function value and first derivative value as $f(x)$ at $x = a$.
As a polynomial of degree n,

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k. \tag{11.11.3}$$

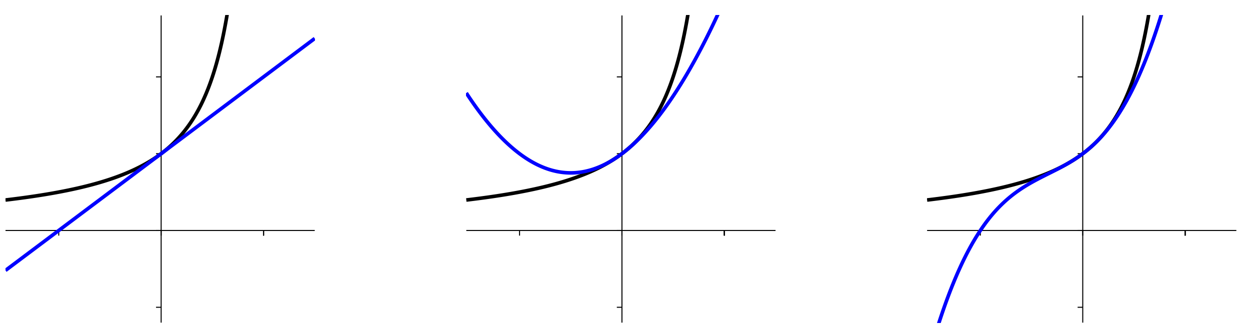
is a power series (with finitely many nonzero terms). By Theorem 11.10.1, its coefficients must be $\frac{1}{k!}T_n^{(k)}(a)$. That is,

$$\frac{T_n^{(k)}(a)}{k!} = \begin{cases} \frac{1}{k!} f^{(k)}(a) & \text{if } k \le n, \text{ and} \\ 0 & \text{if } k > n \end{cases}$$

Consequently, T_n and its first n derivatives match f and *its* first n derivatives at $x = a$:

$$T_n^{(k)}(a) = f^{(k)}(a), \qquad \text{if } k \le n$$

That makes the graph of T_n tangent to the graph of f when $n = 1$, and even "more tangent" when $n > 1$.
For instance, here are the graphs of $\frac{1}{1-x}$ and its Taylor polynomials $T_1(x)$, $T_2(x)$, and $T_3(x)$.



$y = (1-x)^{-1}$ $y = 1 + x$

$y = (1-x)^{-1}$ $y = 1 + x + x^2$

$y = (1-x)^{-1}$ $y = 1 + x + x^2 + x^3$

These figures suggest that the Taylor polynomials for $f(x)$ are approaching $f(x)$ as as $n \to \infty$, and, in fact, this is the case. Is this a consequence of the Taylor polynomials matching the derivatives of f at the point of tangency?
Surprisingly, the answer is no, since there exist functions which possess derivatives of all orders (and therefore possess Taylor polynomials of all degrees) but which are not the limits of their Taylor polynomials. So, how do we know if a function is the limit of these polynomials?

In general, since the Taylor polynomials are the partial sums of the Taylor series, the convergence of the Taylor series to f (if it occurs)

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$$

is equivalent to

$$T_n(x) \to f(x) \text{ as } n \to \infty,$$

which, in turn, is equivalent to

$$\big(f(x) - T_n(x)\big) \to 0 \text{ as } n \to \infty.$$

The difference between $f(x)$ and its Taylor Polynomial $T_n(x)$ is called the nth **remainder** of the Taylor series:

$$R_n(x) = f(x) - T_n(x).$$

So, to determine whether $f(x)$ is the sum of its Taylor series, we have to determine whether $\lim_{n\to\infty} R_n(x) = 0$. Generally, it's not possible to take this limit without somehow rewriting R_n. That's where the next theorem comes in handy.

Taylor's Theorem 11.11.4. *If f and all of its derivatives exist in an interval containing a and x, then*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

Some comments:
1. You should consider the number c in the conclusion as just some unspecified number between x and a. Beyond that, we have no idea what it might be.
2. In case $n = 0$, Taylor's Theorem is just the Mean Value Theorem of Calculus I.
3. $R_n(x)$ is what one would add to $T_n(x)$ to obtain $f(x)$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

11.11.4 says that $R_n(x)$ resembles the next term in the Taylor series, $\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$.
4. Instead of Taylor's theorem, our text includes the following fact.

Taylor's Inequality 11.11.5. *If $|f^{(n+1)}(x)| \le M$ for all x (or for all x within some fixed distance from a) then for all such x,*

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Although it doesn't tell the whole story that Taylor's theorem does, Taylor's inequality includes all we actually need to solve convergence and approximation problems in Calc II.

11.11.e3. Show that, for any a, $\sin x$ is the sum of its Taylor series centered at a.

Error analysis with Taylor's Theorem

Taylor's Theorem is useful when we want to bound the absolute error between f and T_n on an interval about the center a:

$$|f(x) - T_n(x)| \leq B \quad \text{for all } x \text{ in} \quad [a-b, a+b]$$

Typical problems give us two of $\{n, b, B\}$ and ask for the third.

11.11.e4. Bound the absolute error between $f(x) = \ln(1-x)$ and its third Taylor polynomial centered at 0 on the interval $[-0.9, 0.9]$

Tip: Writing out the Taylor polynomial in question usually isn't helpful in error-anaylsis problems.

11.11.e5. Let $f(x) = \cos x$ and $a = 0$. Find b so that the absolute error between f and T_2 is at most 10^{-2} on $[-b, b]$. Tip: $T_2(x) = T_3(x)$ for this function.

11.11.e6. What degree Maclaurin polynomial approximates e^x on the interval $[-1, 1]$ with an absolute error at most 10^{-5}? Hint: see the table.

n	$e/(n+1)!$
2	$4.53E-01$
3	$1.13E-01$
4	$2.27E-02$
5	$3.78E-03$
6	$5.39E-04$
7	$6.74E-05$
8	$7.49E-06$
9	$7.49E-07$

9.3: Seperable Differential Equations

In Calculus I, we used *implicit differentiation* to find $\frac{dy}{dx}$ along a curve.

9.3.e1. Find $\dfrac{dy}{dx}$ along the curve $x^2 + 3y^2 = 1$.

In this chapter, we work in the opposite direction.

9.3.e2. Find all curves along which $2x + 6y\dfrac{dy}{dx} = 0$.

9.3.e2, continued. Which of these curves passes through the point $(-2, 1)$?

Definition 9.3.1. *A (1st order)* **differential equation** *is an equation in x, y, and $\frac{dy}{dx}$.*

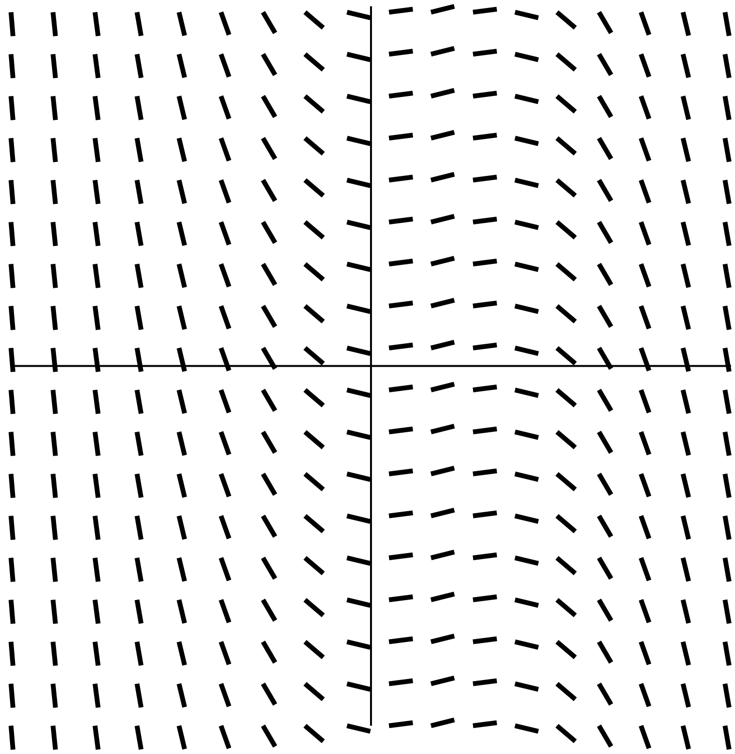
The **general solution** *of a differential equation is the family of all functions y that satisfy the differential equation, or, more generally, the family of all equations in x and y along which the differential equation is satisfied.*

An **initial value problem** *is a differential equation plus a point (x_0, y_0). Its solution is the particular member of this family that passes through the given point.*

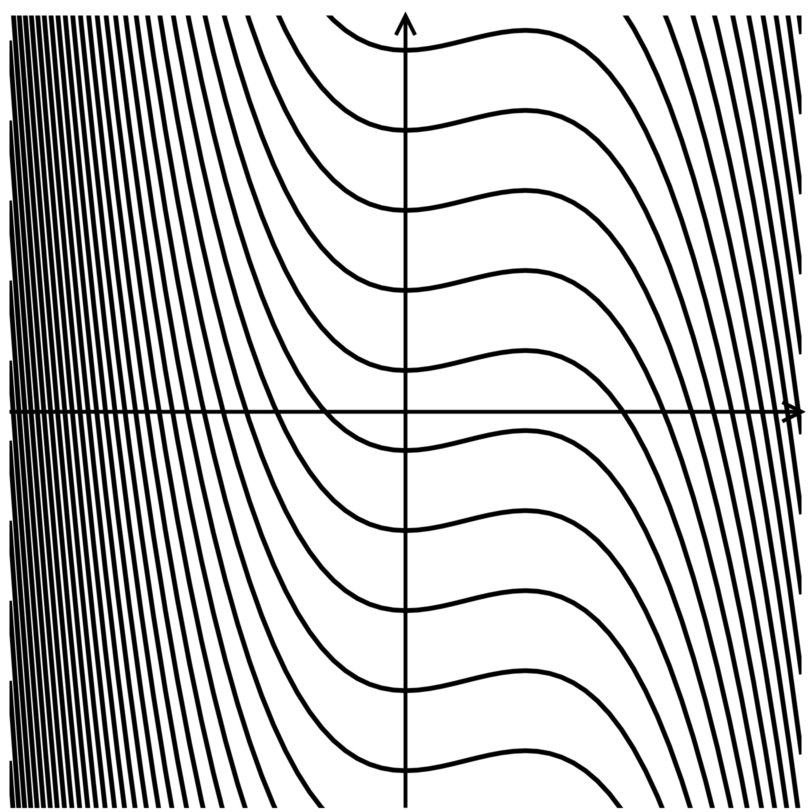
9.3.e3. Find the general solution to the differential equation $\frac{dy}{dx} = x - x^2$ and the particular solution that also satisfies $y = 1$ when $x = -2$.

9.3.e3, continued.

$$\frac{dy}{dx} = x - x^2$$

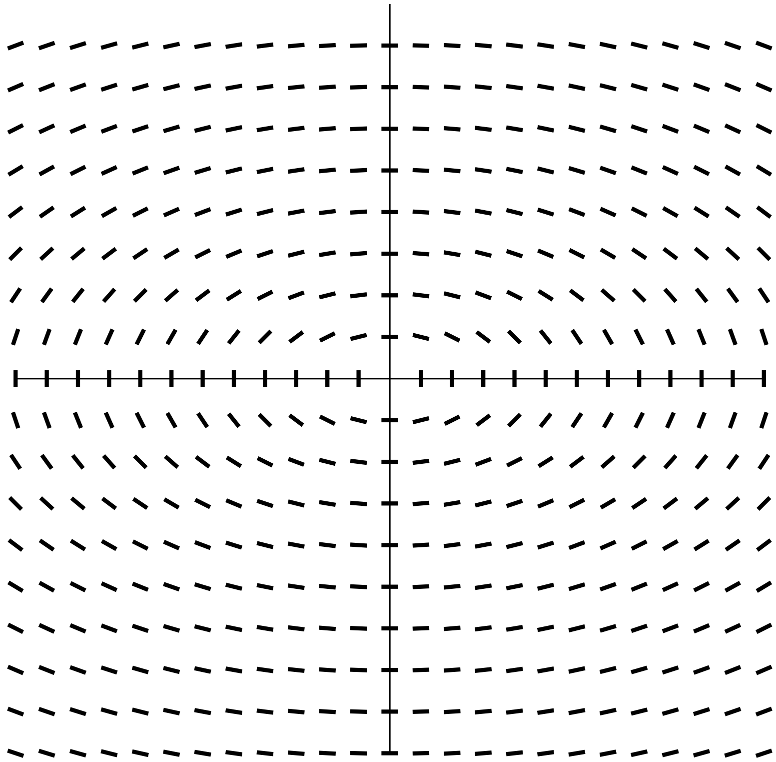


$$y = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + C$$

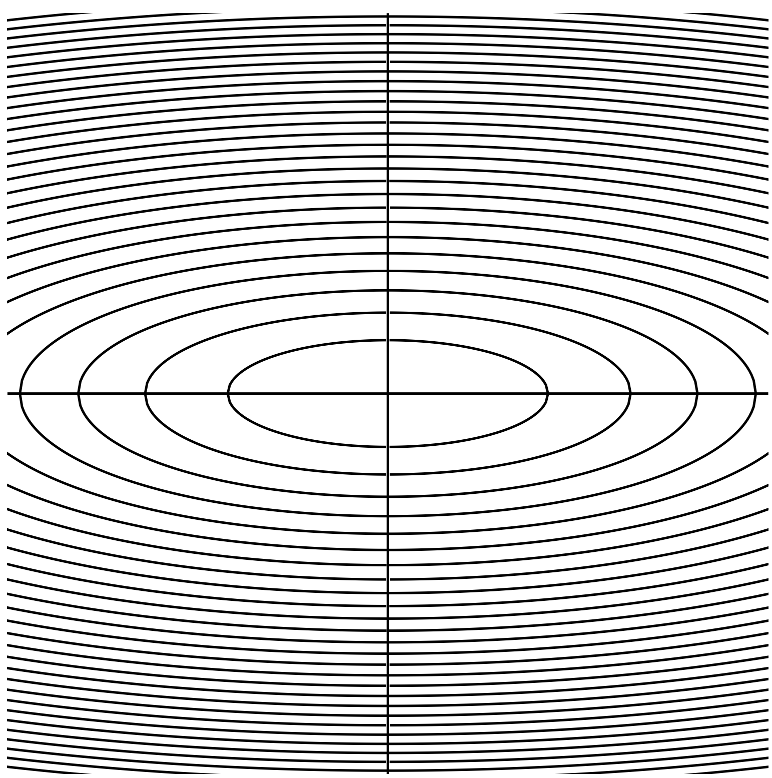


9.3.e2, continued.

$$2x + 6y\frac{dy}{dx} = 0$$



$$x^2 + 3y^2 = C$$



In 9.3, we study **separable** differential equations, i.e., those which can be written

$$f(x)\,dx = g(y)\,dy$$

To solve a separable equation,

1. **Separate variables**
2. **Integrate**

9.3.e4. a. Solve the differential equation. $\dfrac{dy}{dx} + xy^2 = 0.$

Note: To "solve" a differential equation always means to find its general solution.

9.3.e4 b. Find particular solution to 9.3.e4a that passes through $(1, -2)$.

9.3.e5. Solve: $\dfrac{dy}{dx} - \dfrac{x}{y}e^{x-y} = 0$, with $y = 1$ at $x = 2$.

Examples of differential equations in natural sciences

9.3.e6. Population with a constant percent rate of growth:

$$\frac{dy}{dt} = \frac{1}{5}y$$

Exponential Growth

The general solution to $\dfrac{dy}{dt} = ky$ *is* $y = y_0 e^{kt}$.

9.3.e7. Population growth with maximum sustainable population :

$$\frac{dy}{dt} = \frac{1}{5}y\left(1 - \frac{y}{500}\right)$$

Logistic Growth

9.3.e8. Radioactive decay:

$$\frac{dy}{dt} = -\frac{1}{100}y$$

Exponential Decay

Orthogonal Trajectories

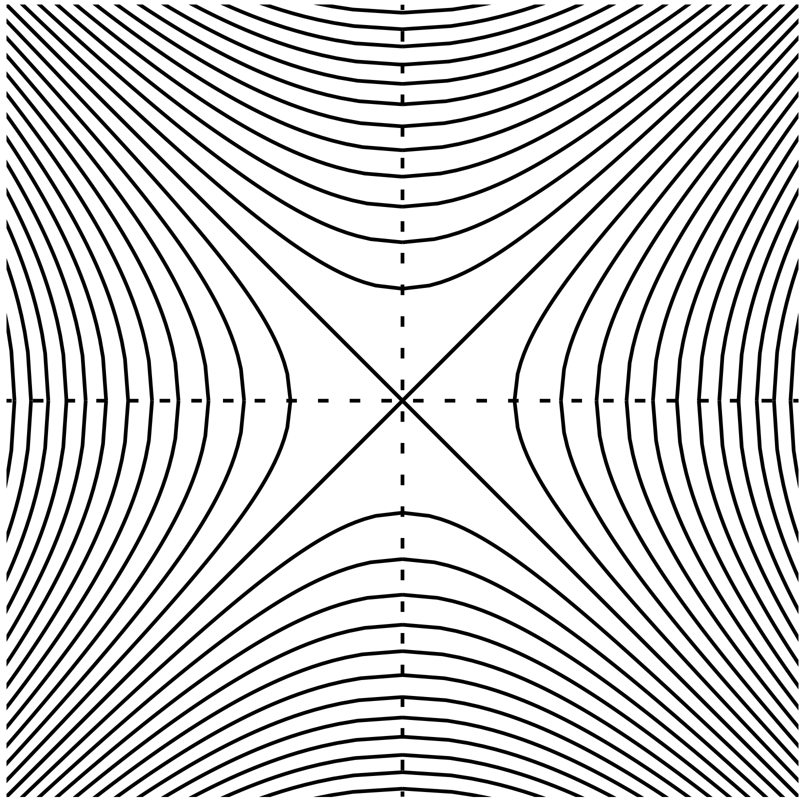
9.3.e9. Find the orthogonal trajectories of the curves $x^2 - y^2 = C$.

For more orthogonal trajectories exercises, see Problem 1 on
http://kunklet.people.cofc.edu/MATH220/stew0903prob.pdf
(There's a link to this file in the assigned problems from 9.3 on our syllabus, under more.)

9.3.e9, continued.

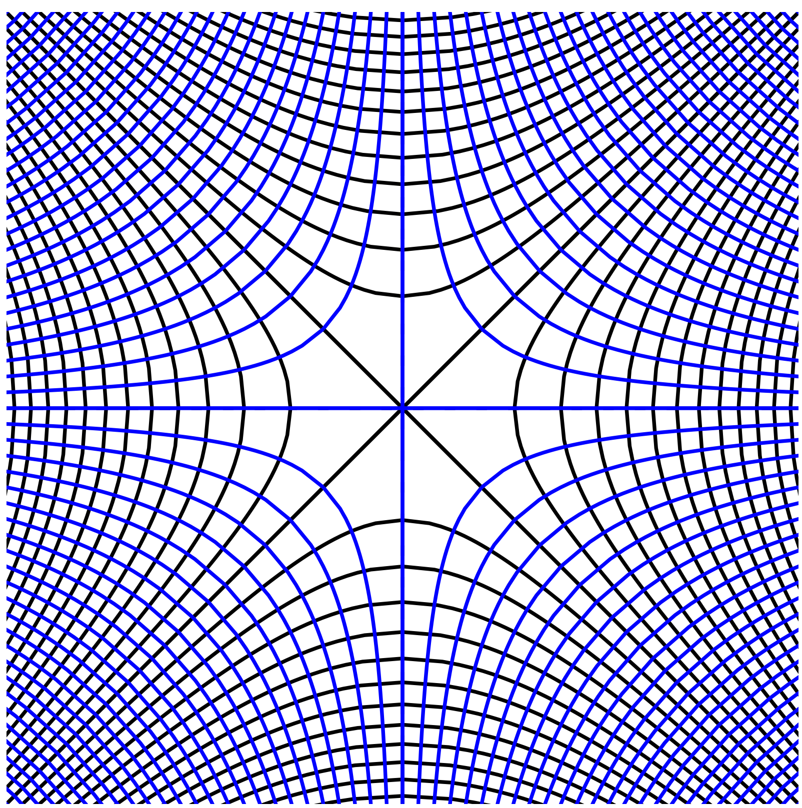
$x^2 - y^2 = C$

$$\frac{dy}{dx} = \frac{x}{y}$$



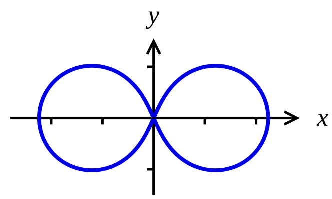
$$\frac{dy}{dx} = -\frac{y}{x}$$

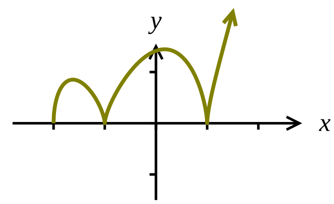
$xy = C$



10.1: Parametric Equations of Curves

In calculus so far, we've represented curves in the plane **implicitly**, that is, as solutions to xy-equations, sometimes an equation in the special form $y = f(x)$.





10.1.e1. $(x^2+y^2)^2 = 5x^2 - y^2$ $\quad$ $y = (x+2)^{1/2}(1-x^2)^{2/3}$

But, producing a curve from its implicit representation is numerically challenging. For instance, the two equations

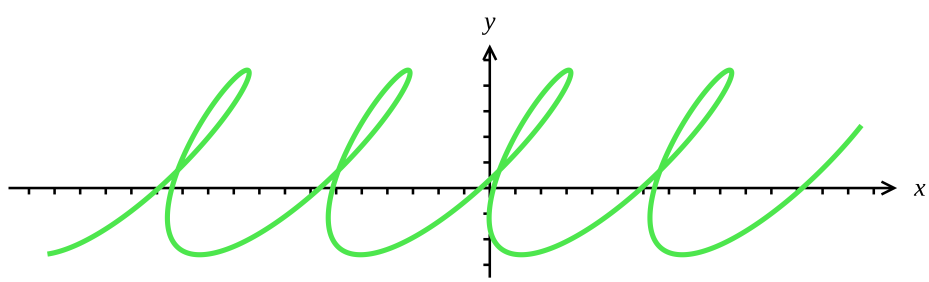
$$(x+y)(y-x^2) = 0$$

and

$$(x+y)^2(y-x^2) = 0$$

both have the same graph, consisting of the line $y = -x$ (where the first factor of each is zero) and the parabola $y = x^2$ (where the second factor is zero). But compare their graphs produced by Desmos at https://www.desmos.com/calculator/x2yoabmkeg.

Another useful way to express curves is **parametrically**, that is, by setting expressing x and y as functions of a parameter t, say $x(t)$ and $y(t)$, and then tracing out the path of a particle whose position at time t is $\big(x(t), y(t)\big)$.



10.1.e2. $x = t + 3\cos t,\ y = 1 + 3\cos t + 2\sin t$

Parametrization is the natural way to describe to motion of an object in the plane or, if we add a third function $z(t)$, space.
Parametric equations of curves (and surfaces, seen in Calculus III) are far easier to graph than implicit equations and for this reason are preferred applied curve and surface fitting.

Deriving an implicit representation of a curve from its parametric representation—when that's even possible—is generally referred to as **elimination of the parameter**.

10.1.e3. Sketch the parametrized curve, indicating the direction the curve is traced with arrows. Eliminate the parameter.

a. $x = \sin t$, $y = \cos t$

b. $x = 2 + \sin t$, $y = \frac{1}{2} + \cos t$

c. $x = 2 \sin t$, $y = \frac{1}{2} \cos t$

10.1.e4. Even when we manage to eliminate the parameter t, the xy equation might include points not covered by the parametrized curve:

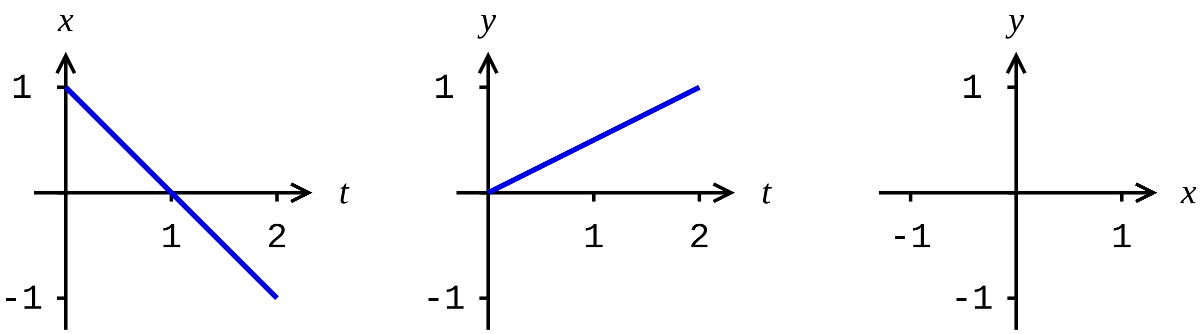
a. $x = t + 2$, $y = 1 + t^2$

b. $x = t^2 + 2$, $y = 1 + t^4$

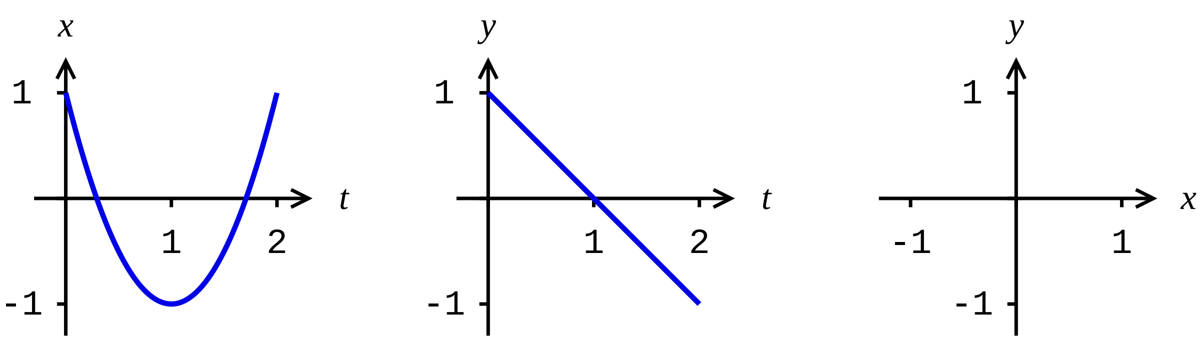
c. $x = \sin^2 t$, $y = \cos^2 t$

10.1.e5. Use the graphs of $x(t)$ and $y(t)$ to sketch the curve given parametrically by $x = x(t)$, $y = y(t)$. Indicate direction with arrows.

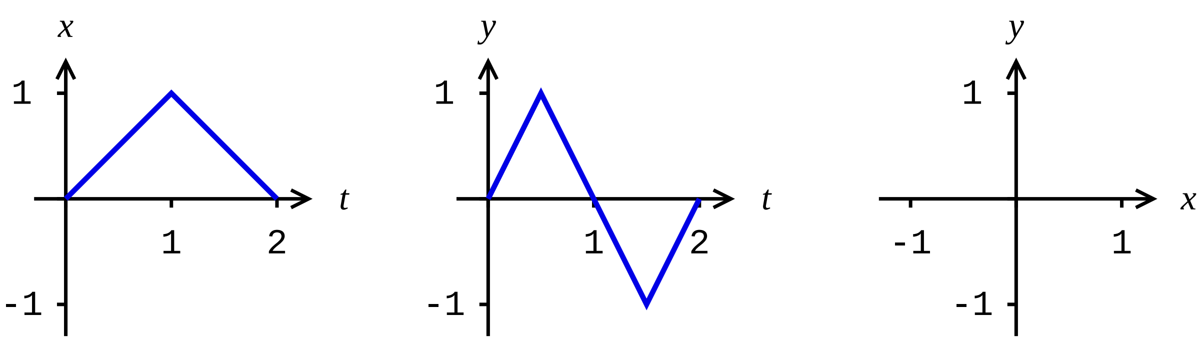
a.



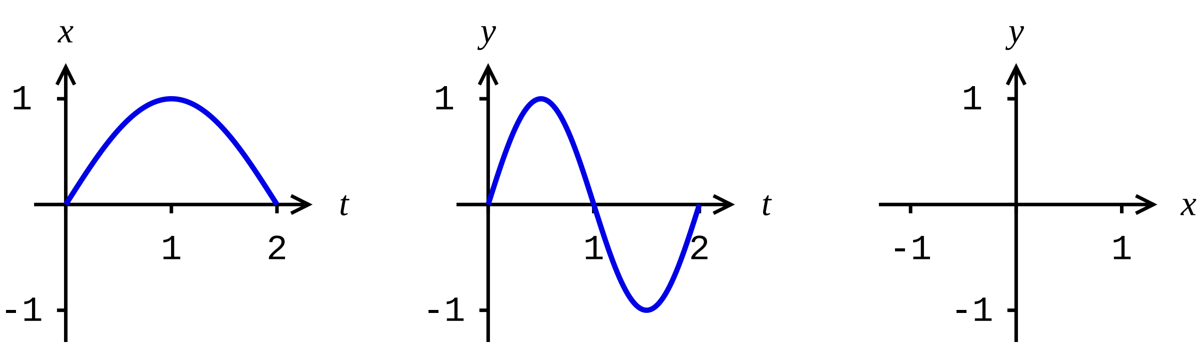
b.



c.



d.



10.2: Calculus on Parametrized Curves

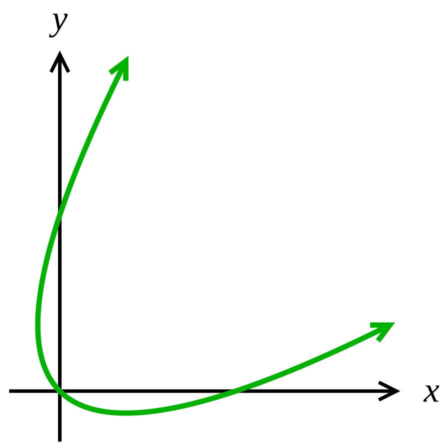
10.2.e1. Sketch the graph of $x = t^2 - 2t$, $y = t^2 + 2t$.

Calculating $\dfrac{dy}{dx}$ *along a parametrized curve*

To find $\frac{dy}{dx}$ from the functions $x(t)$ and $y(t)$, use the chain rule: $\frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dt}$ implies

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}}$$

10.2.e2. Find $\dfrac{dy}{dx}$ along the curve $x = t^2 - 2t$, $y = t^2 + 2t$ from 10.2.e1.

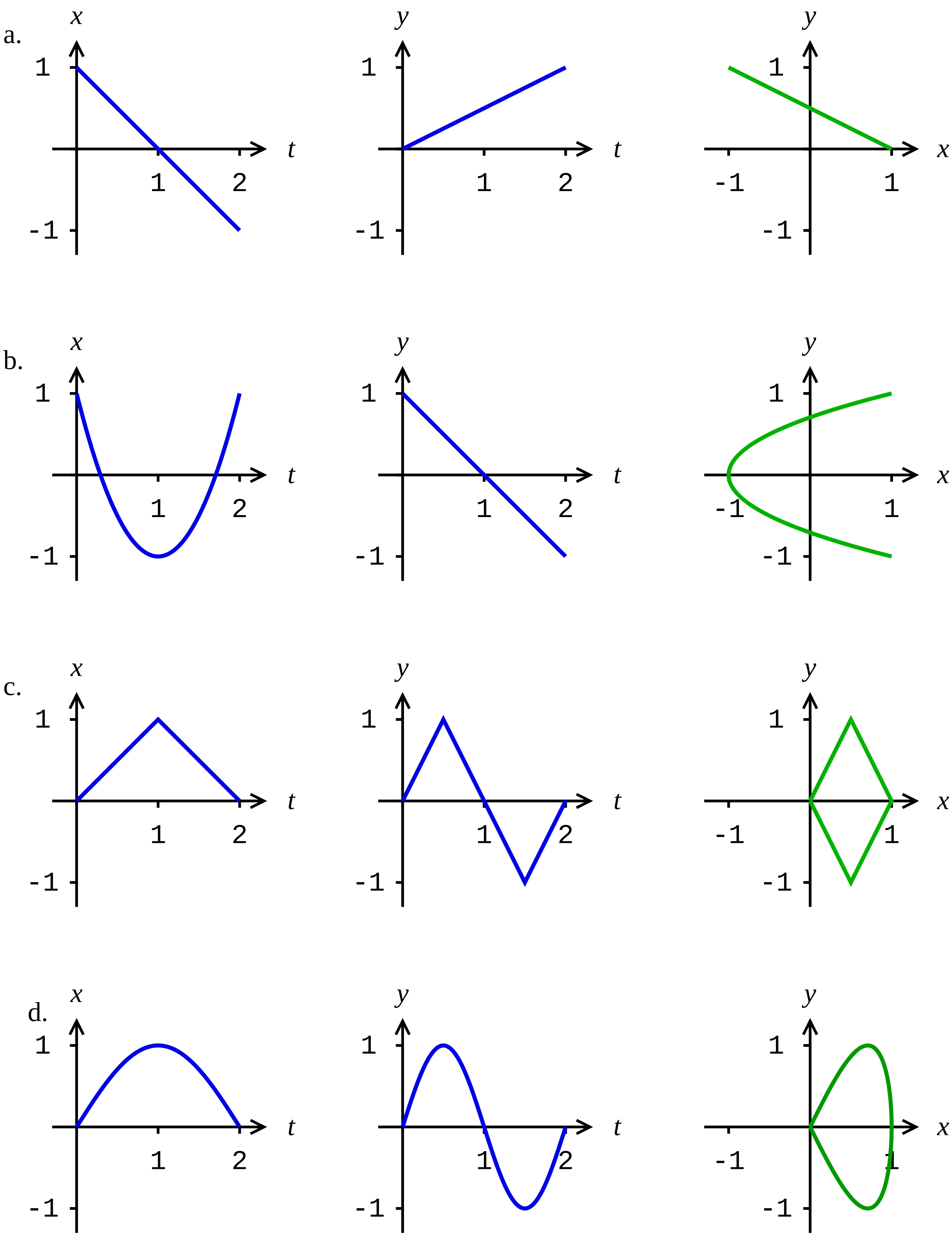


As a rule,

When $\dfrac{dx}{dt} \neq 0 = \dfrac{dy}{dt}$, the curve has a **horizontal** tangent line.

When $\dfrac{dx}{dt} = 0 \neq \dfrac{dy}{dt}$, the curve has a **vertical** tangent line.

10.2.e3. Here are the curves we sketched by hand in 10.1.e5.



Calculating $\dfrac{d^2y}{dx^2}$ *along a parametrized curve*

To find the second derivative of y with respect to x, use the same principal as before:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \quad \Longrightarrow \quad \frac{d}{dx}(\text{Anything}) = \frac{\dfrac{d}{dt}(\text{Anything})}{\dfrac{dx}{dt}}$$

so

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}}$$

10.2.e4. Find $\dfrac{d^2y}{dx^2}$ along the curve $x = t^2 - 2t$, $y = t^2 + 2t$ from 10.2.e1.

Arclength of a parametrized curve

As in section 8.1, differential arclength $ds = \sqrt{(dx)^2 + (dy)^2}$. When x and y are functions of t, factor dt out of the radical to get a useable expression for ds:

$$\boxed{ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

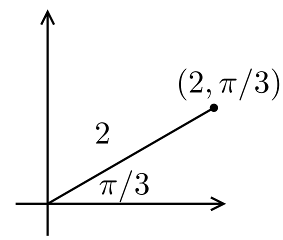
10.2.e5. Find the length of the curve $x = \sqrt{3}t^2 - 1$, $y = t - t^3$ for $0 \le t \le 1$.

10.3: Polar coordinates

A coordinate system is a way of giving directions, in form of a pair of numbers, to a point in the plane. For instance, the rectangular coordinates (x, y) direct us to the point x units to the right (left, if $x < 0$) and y units up (down, if $y < 0$) from the origin.

The polar coordinate system gives direction to a point using two numbers r and θ. To reach the point, stand at the origin, facing the direction of the positive x-axis. Turn θ radians counterclockwise (clockwise, if $\theta < 0$), then move r units forward (backward, if $r < 0$). (Counterclockwise and clockwise are more simply referred to as the the **positive** and **negative** directions, respectively.)

For instance, the polar coordinates $(2, \pi/3)$ direct us to the point in the first quadrant two units from the origin along the ray that is $\pi/3$ radians from the x-axis:



10.3.e1. Plot the points with the following polar coordinates.

a. $(2, 3\pi/4)$

b. $(2, -3\pi/4)$

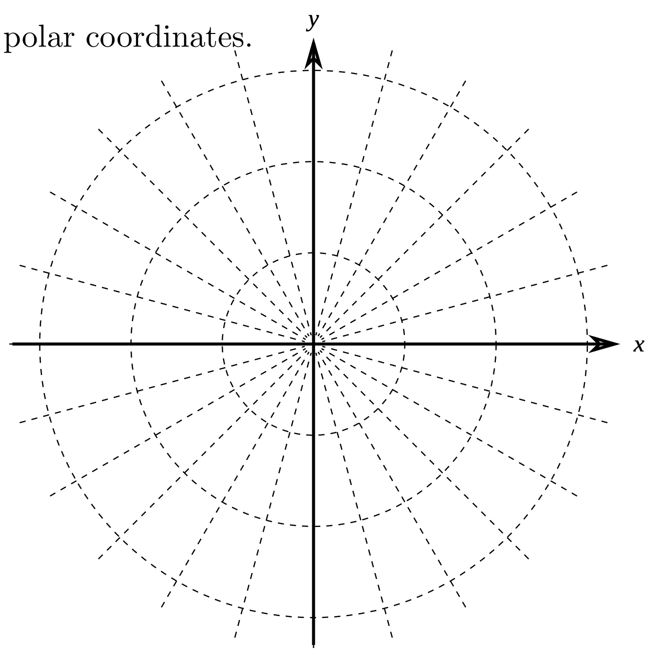
c. $(2, -5\pi/4)$

d. $(3, 7\pi/6)$

e. $(-3, \pi/6)$

f. $(0, 5\pi/6)$

g. $(0, -1)$



In general,

$$(r, \theta), \quad (r, \theta + 2\pi), \quad (-r, \theta + \pi)$$

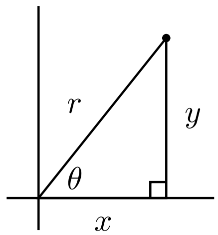
all represent the same point in the plane, and therefore so do

$$(r, \theta + 2n\pi), \quad (-r, \theta + (2n + 1)\pi)$$

for any integer n. If $r \neq 0$, no other polar coordinates describe the same point as these. (If $r = 0$, then the point is the origin regardless of θ.)

Polar vs. Rectangular coordinates

Practice writing these four relationships between (r, θ) and (x, y) with the help of this picture (drawn as if r were positive and θ were acute):



10.3.e2. Covert the equation from one coordinate system to the other and identify its graph.

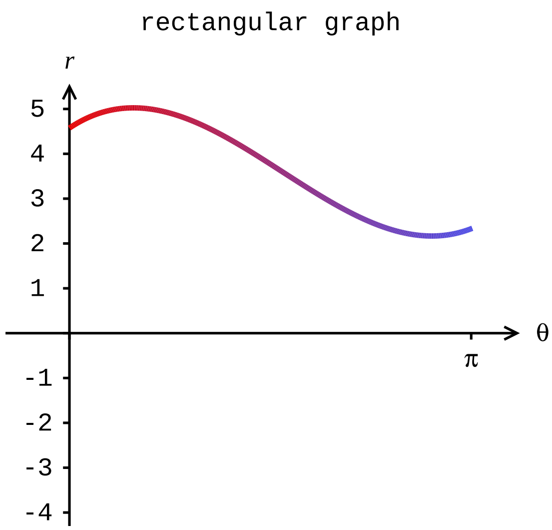
a. $\theta = \pi/6$

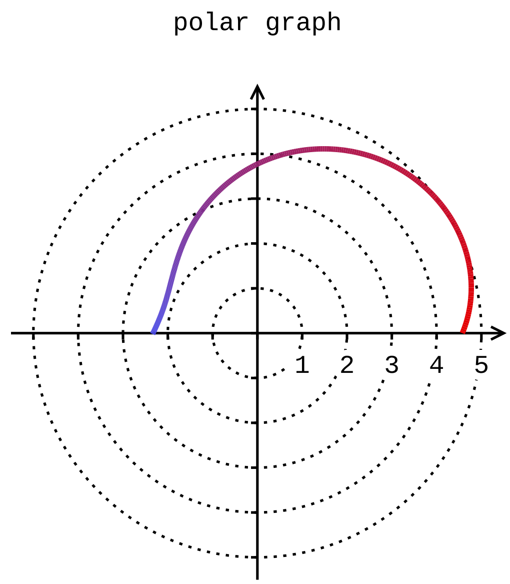
b. $x^2 + y^2 = 4$

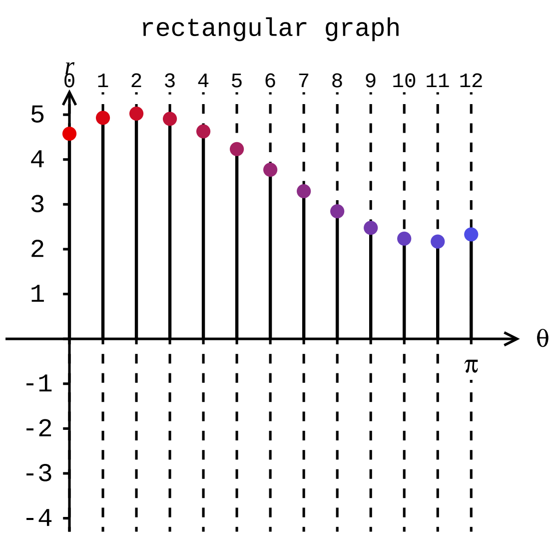
c. $r = 2\cos\theta$

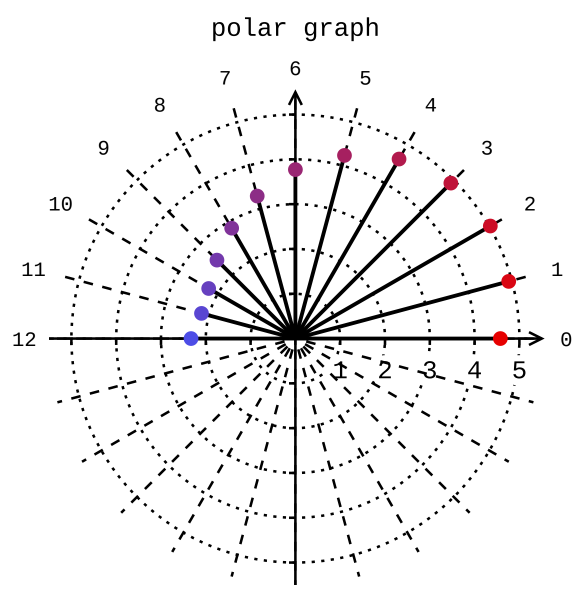
Some general advice for graphing in polar coordinates

1. Most of the polar equations we need to graph are of the form $r = r(\theta)$. To graph a polar equation, it helps to first graph it as a *rectangular* equation with θ on the horizontal and r on the vertical.

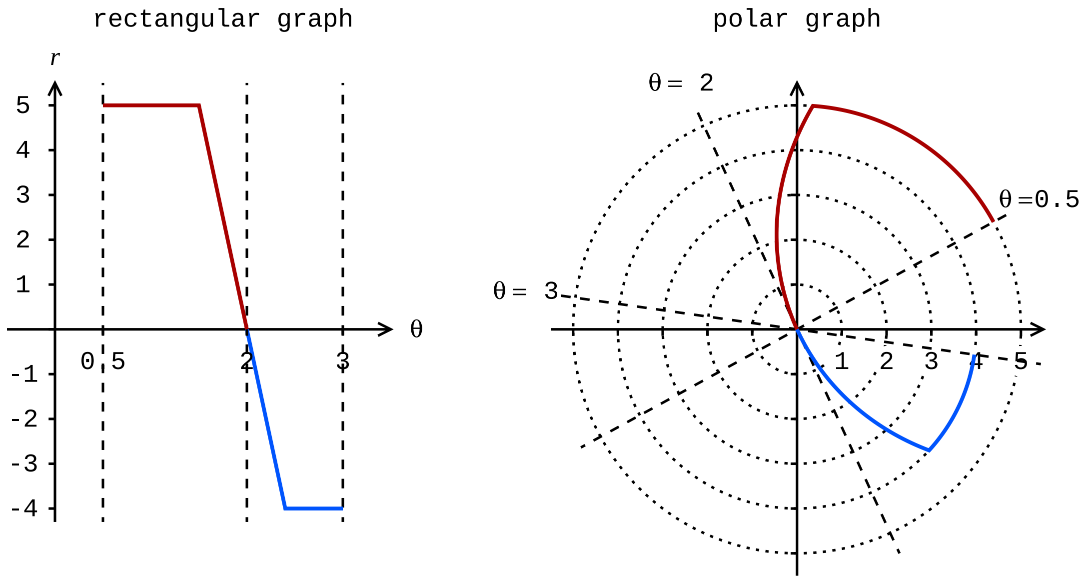






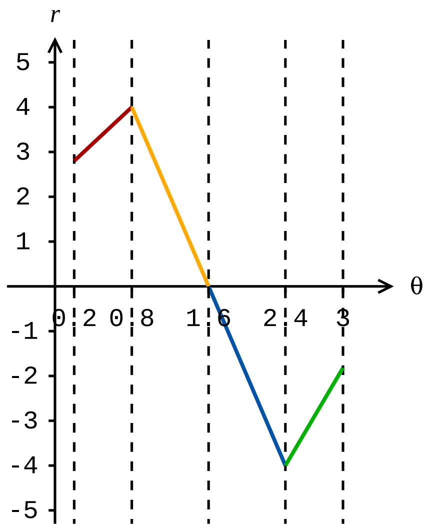


2. If $r(\theta) > 0$ for $\theta_0 < \theta < \theta_1$, the graph of $r(\theta)$ appears between the rays at θ_0 and θ_1.
   If $r(\theta) < 0$ for $\theta_0 < \theta < \theta_1$, the graph of $r(\theta)$ appears between the rays at θ_0 and θ_1 on the opposite side of the origin.
   If $r = 0$ at θ_0, then the curve passes through the origin tangent to the ray $\theta = \theta_0$.

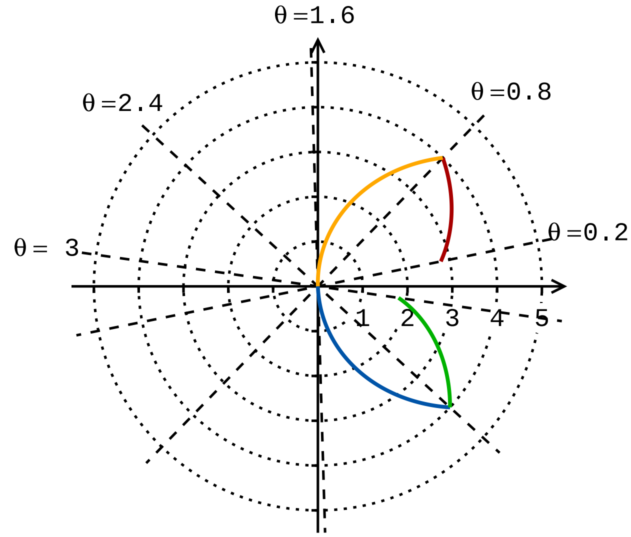


3. Whenever r is getting closer to zero, the graph of $r = r(\theta)$ is spiraling in towards the the origin, and whenever r is getting farther from zero, $r = r(\theta)$ is spiraling out away from the origin.

rectangular graph

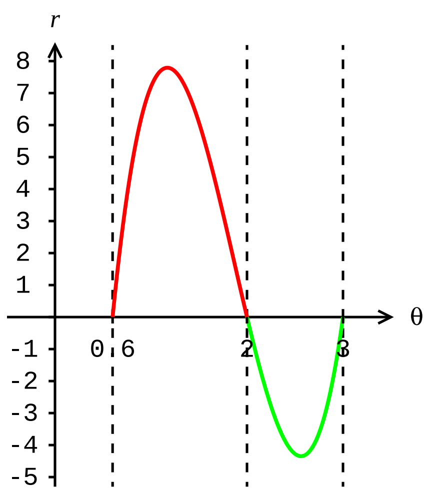


polar graph

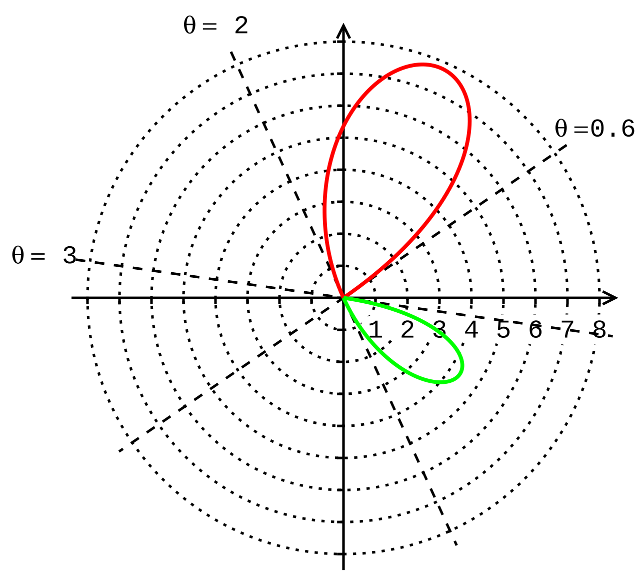


4. Between two consecutive zeros θ_0 and θ_1 of the function r, the graph of $r = r(\theta)$ forms a loop, (sometimes called a **petal**) that begins and ends at the origin. If $r \geq 0$, the loop appears between the rays at θ_0 and θ_1. If $r \leq 0$, the loop appears between the rays at θ_0 and θ_1 on the opposite side of the origin.

rectangular graph

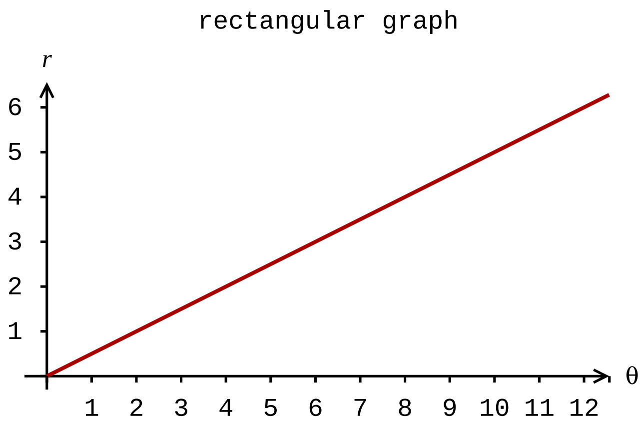


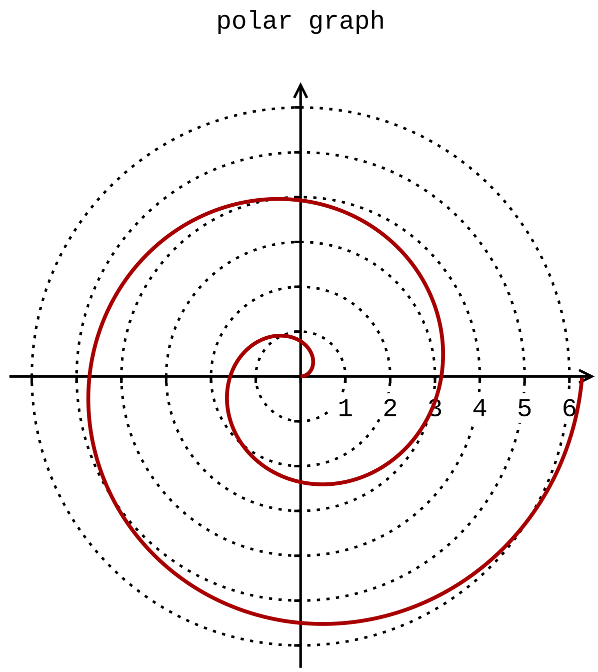
polar graph



Spirals: $r(\theta)$ positive and either increasing or decreasing

10.3.e3. Graph the polar equation $r = \frac{1}{2}\theta$ $(\theta \geq 0)$.

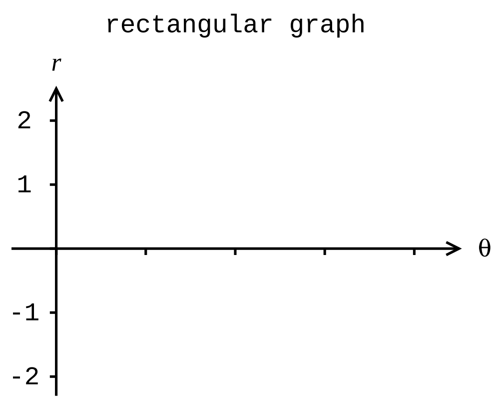


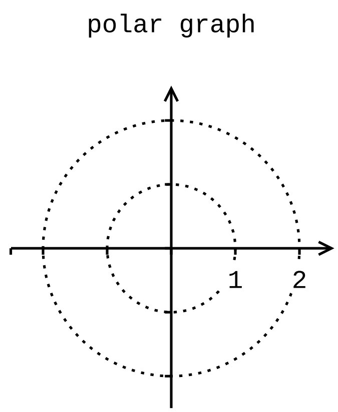


10.3.e4. Graph the polar equation $r = 3e^{-\theta}$.

Circles: $r = a\sin\theta$, $r = a\cos\theta$, or $r =$ constant

10.3.e5. Graph the polar equation $r = 2\cos\theta$.



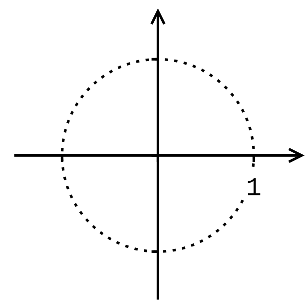


Roses: $r = a\cos(n\theta)$ *and* $r = a\sin(n\theta)$

10.3.e6. Graph the polar equation $r = \sin(2\theta)$

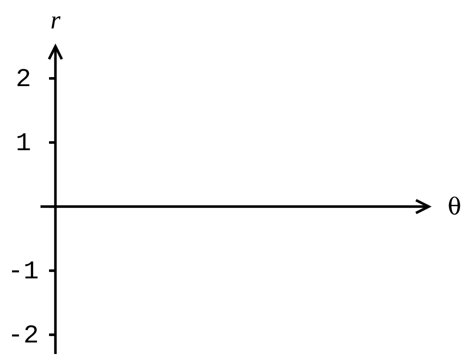
rectangular graph

polar graph

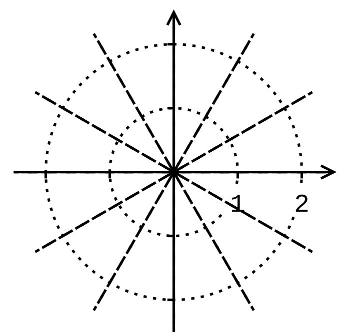


10.3.e7. Graph the polar equation $r = 2\cos(3\theta)$

rectangular graph

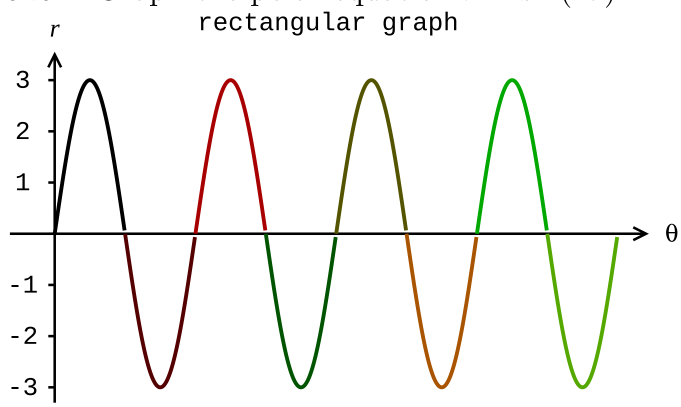


polar graph

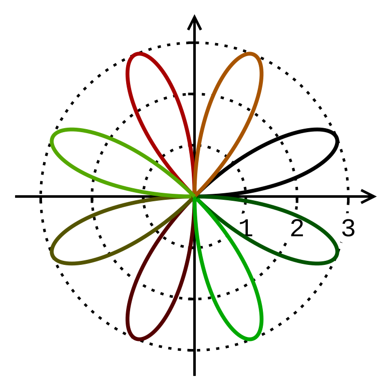


10.3.e8. Graph the polar equation $r = \sin(4\theta)$

rectangular graph



polar graph



Roses are composed of petals. The length of each petal from the origin to its farthest point is $|a|$, and the number of petals on the rose is

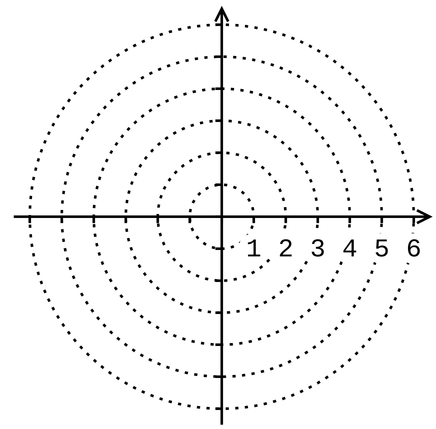
$$\begin{cases} n & \text{if } n \text{ is odd, and} \\ 2n & \text{if } n \text{ is even.} \end{cases}$$

To graph more roses, go to

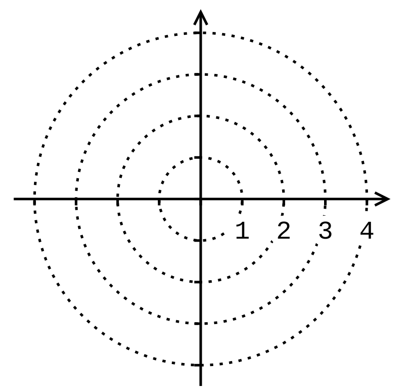
https://www.desmos.com/calculator/zxgo9rhhzv

Limaçons: $r = b + a\cos\theta$ *and* $r = b + a\sin\theta$

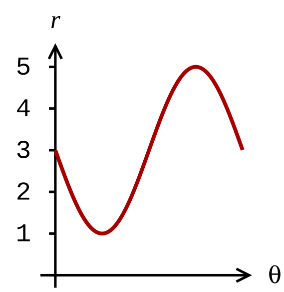
10.3.e9. Graph the polar equation $r = 3 + 3\cos\theta$

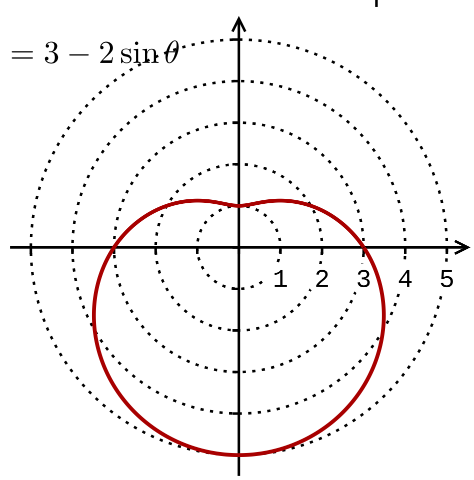


10.3.e10. Graph the polar equation $r = 1 + 3\sin\theta$



10.3.e11. Graph the polar equation $r = 3 - 2\sin\theta$





A limaçon's shape depends on the $a \pm b$, maximum and minimum values of r. The graph touches the origin if $r = 0$ for some θ, and the graph contains an inner loop if and only if r changes sign (that is, if its maximum $> 0 >$ its minimum.)

For an interactive graph of limaçons, visit
https://www.desmos.com/calculator/yr4rfqf5eh

Summary of graphing polar equations

You should be comfortable graphing the following polar curves:

Spirals	$r > 0$ an increasing or decreasing function
Circles	$r =$ constant, $r = a\sin\theta$, $r = a\cos\theta$
Roses	$r = a\cos(n\theta)$ or $r = a\sin(n\theta)$
Limaçons	$r = b + a\cos\theta$ or $r = b + a\sin\theta$

Slope along a polar curve

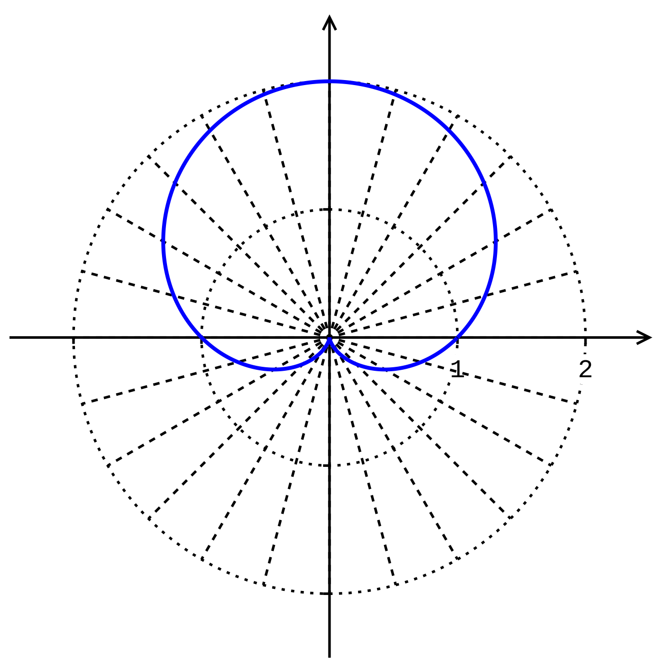
We don't need a special formula to find $\dfrac{dy}{dx}$ along a polar curve of the form $r = r(\theta)$, because such a curve is given parametrically by

$$x = r(\theta)\cos\theta \qquad y = r(\theta)\sin\theta,$$

and therefore

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

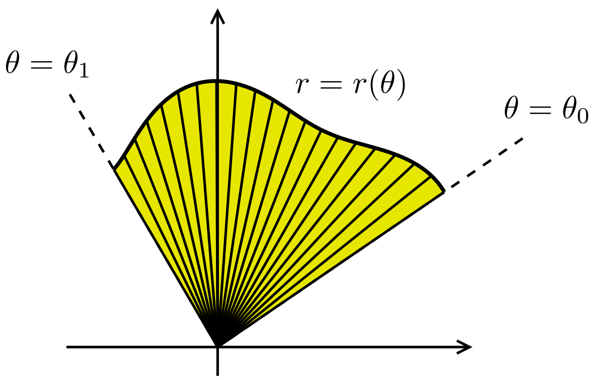
10.3.e12. Find $\dfrac{dy}{dx}$ along the curve $r = 1 + \sin\theta$

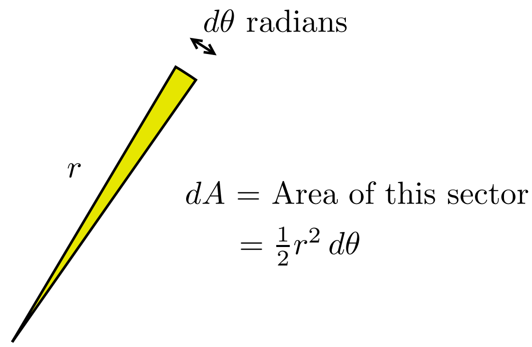


10.4: Area and arclength in polar coordinates

Area inside a polar curve

We wish to find the area trapped by a polar curve $r = r(\theta)$ between two rays, $\theta_0 \le \theta \le \theta_1$. Note that this is *not* the same as the area between the curve and the x-axis.
Slice the region up with cuts along every ray through the origin between θ_0 and θ_1, and let $d\theta$ be the infinitesimally small difference between the values of θ at the two sides of the slice. Then the total area $A = \int dA$, where dA is area of the the infinitesimally small slice at θ. Because θ changes so little along this slice, r is essentially constant, and the slice is





a sector of a circle. The sector contains $d\theta/2\pi$ of an entire circle of radius r, so its area is

$$dA = \pi r^2 \left(\frac{d\theta}{2\pi}\right) = \frac{1}{2} r^2\,d\theta.$$

Area problems in polar coordinates often require the use of the half-angle formulas

$$\cos^2 x = \tfrac{1}{2}\bigl(1 + \cos(2x)\bigr) \qquad \sin^2 x = \tfrac{1}{2}\bigl(1 - \cos(2x)\bigr),$$

which can obtained by adding and subtracting

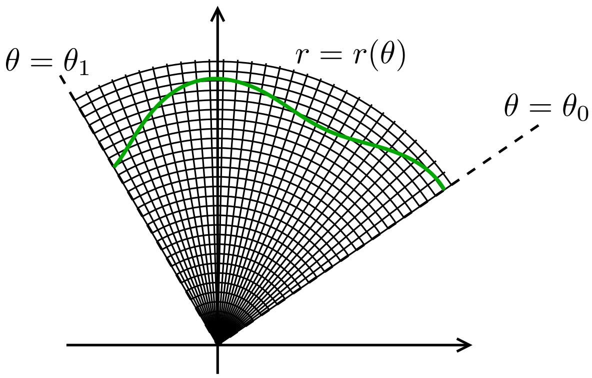
$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ 1 &= \cos^2 x + \sin^2 x \end{aligned}$$

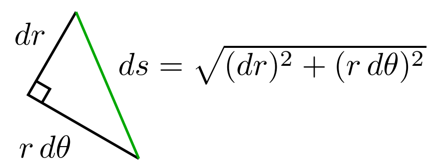
10.4.e1. Find the area inside one loop of $r = \sin(4\theta)$.

10.4.e2. Find the area inside $r = 1 + 2\cos\theta$ and outside $r = 2$. Express your answer as a definite integral, but do not evaluate.

Arclength of a polar curve

To find the length of a polar curve $r = r(\theta)$ over $\theta_0 \le \theta \le \theta_1$, chop the curve into infinitely many pieces, each of which is infinitesimally small, and let ds be the length of the piece of curve at the point (r, θ). Along this piece, there's a change of dr units along a ray (where θ is constant), and $r\,d\theta$ units along a circle (where r is constant). Since the rays and the





circles are orthogonal trajectories, we can use the pythagorean theorem to find

$$ds = \sqrt{(dr)^2 + (r\,d\theta)^2}$$

Factor out the positive $d\theta$ to obtain

$$ds = \left(\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\right) d\theta$$

.

10.4.e3. Find the length of one loop of the rose $r = \cos 2\theta$. Write your answer as a definite integral but do not integrate.