MATH 203–03 (Kunkle), Quiz 7 Name: ____________________

10 pts, Take-home Due 9:00am, Oct 25, 2024

1 (10 pts). This question has several parts. To avoid unncecessary duplication of work, read the entire question before begining your solution.

a. Find bases for the Col W, Row W, and Nul W if

$$W = \begin{bmatrix} 0 & 1 & 2 & -3 \\ 1 & -2 & -5 & 8 \\ 3 & -4 & -11 & 18 \end{bmatrix}$$

b. Determine whether $\mathbf{b} = [5 \quad 3 \quad 20]^T$ is in Col W, and if it is, find its coordinates relative to the basis for Col W that you found in part a.

c. Determine whether $\mathbf{c} = [-2 \quad 6 \quad 14]^T$ is in Col W, and if it is, find its coordinates relative to the basis for Col W that you found in part a.

Solution:

1(10 pts). To answer all three parts, we'll need row echelon forms of W, of the augemented matrix $[W\ \mathbf{b}]$, and of $[W\ \mathbf{c}]$. To save work, let's augment both $\mathbf{b}$ and $\mathbf{c}$ to W and row reduce.

Forward phase:

row operation	result
initial matrix	$\begin{matrix} 0 & 1 & 2 & -3 & 5 & -2 \\ 1 & -2 & -5 & 8 & 3 & 6 \\ 3 & -4 & -11 & 18 & 20 & 14 \end{matrix}$
$\mathbf{r}_2 \rightleftarrows \mathbf{r}_1$	$\begin{matrix} 1 & -2 & -5 & 8 & 3 & 6 \\ 0 & 1 & 2 & -3 & 5 & -2 \\ 3 & -4 & -11 & 18 & 20 & 14 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_1$	$\begin{matrix} 1 & -2 & -5 & 8 & 3 & 6 \\ 0 & 1 & 2 & -3 & 5 & -2 \\ 0 & 2 & 4 & -6 & 11 & -4 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_2$	$\begin{matrix} 1 & -2 & -5 & 8 & 3 & 6 \\ 0 & 1 & 2 & -3 & 5 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$

End forward phase.

Augmented matrix is in row echelon form, and we're ready to answer part b:

b. There's a pivot in the augmented Red column, so $W\mathbf{x} = \mathbf{b}$ is inconsistant. That is, $\mathbf{b}$ is not in Col W.

Moving on, let's drop the Red column and proceed with the backward phase.

row operation	result
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + 2\mathbf{r}_2$	$\begin{matrix} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$

To answer part a., ignore the Blue column.

a. The pivot columns of W form a basis for $\operatorname{Col} W$:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} \right\} \text{ is a basis for } \operatorname{Col} W.$$

The nonzero rows of *any* row echelon form of W form a basis for $\operatorname{Row} W$. Using the reduced row echelon form,

$$\{ [1 \quad 0 \quad -1 \quad 2], [0 \quad 1 \quad 2 \quad -3] \} \text{ is a basis for } \operatorname{Row} W.$$

To find a basis for $\operatorname{Nul} W$, find the general solution to $W\mathbf{x} = \mathbf{0}$: x_3 and x_4 are free, and any $\mathbf{x}$ in $\operatorname{Nul} W$ can be written

$$\mathbf{x} = \begin{bmatrix} x_3 - 2x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \text{ so}$$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \operatorname{Nul} W.$$

Let's name this basis $\mathcal{B}$ so we can refer to it in our solution to part c:

c. Note that $W\mathbf{x} = \mathbf{c}$ is consistent, since

$$\begin{array}{ccccc} 1 & 0 & -1 & 2 & 2 \\ 0 & 1 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

does not have a pivot in the augmented column. To express $\mathbf{c}$ as a linear combination of elements of $\mathcal{B}$, find the solution of $W\mathbf{x} = \mathbf{c}$ with the free variables equal zero, that is, the solution to this system:

$$\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}$$

So, $x_1 = 2$ and $x_2 = -2$, and the coordinates of $\mathbf{c}$ are $[\mathbf{c}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

Here are some helpful tips about Octave, including how to use it to calculate the reduced row echelon form: https://bit.ly/3Us8rGp