MATH 203–03 (Kunkle), Quiz 3 | Name: ____________________
10 pts, 10 minutes | Sept 10, 2024

1a (8 pts). Define the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the rule

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & -1 & -4 \end{bmatrix} \mathbf{x}$$

Find a vector $\mathbf{x}$ for which $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$. Is there more than one such $\mathbf{x}$?

1b (2 pts). Based on your work in 1a, would you say that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} \right\}$$

is linearly independent?

Solution:

1a.(Source: 1.8.3-4) This question is about the solutions to a linear system. Here's row elimination on the augmented matrix:

row operation	result
initial matrix	$\begin{matrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 2 & 5 \\ 1 & -1 & -4 & 0 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1$	$\begin{matrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{matrix}$

row operation	result
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_2$	$\begin{matrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_2$	$\begin{matrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$

$\mathbf{x}$ is not unique, since there is a free variable x_3. If $x_3 = 0$, say, then $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

1b.(Source: 1.7.1-2) The above shows that the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 2 \\ 1 & -1 & -4 \end{bmatrix}$$

fails to have a pivot in every column. This means that $T(\mathbf{x}) = \mathbf{0}$ has solutions other than the trivial solution $\mathbf{x} = \mathbf{0}$. That is, the columns of this matrix are linearly dependent.