No notes, books, electronic devices, or outside materials of any kind.
Read each problem carefully and simplify your answers.
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.
Solve or **find the solution** always means to find the general solution, if it exists.
You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1a(18 pts). Compute the determinant of $F = \begin{bmatrix} -1 & 8 & 2 & 4 \\ 0 & 3 & -4 & -4 \\ 1 & -1 & 5 & 5 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

1b(2 pts). Based on your answer to part a, would you say that F invertible? Why?

2. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^5 : \mathbf{x} \mapsto A\mathbf{x}$ for some matrix A.
(That is, T is a transformation from $\mathbb{R}^4$ into $\mathbb{R}^5$ is given by the rule $T(\mathbf{x}) = A\mathbf{x}$.)

a(2 pts). What is the size of A? (That is, how many rows and columns does A have?)

b(5 pts). What would it mean for a transformation S to map $\mathbb{R}^n$ *onto* $\mathbb{R}^5$? In a single sentence, give the definition or an equivalent statement.

c(5 pts). Could T map $\mathbb{R}^4$ onto $\mathbb{R}^5$? Why or why not?

3(6 pts). Suppose that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, and $\mathbf{u}_4$ are four vectors in $\mathbb{R}^4$, and that the equation $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ has no solution.
How many solutions are there to $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 = \mathbf{0}$, and why?

4(12 pts). Answer either a. or b. Clearly indicate which part you're answering.

a. Let S be the transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_1x_2 \\ 2x_2 + x_1x_2 \end{bmatrix}$.
Show that S is *not* linear.

b. Let H be the subset of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy $x_1x_2 - 2x_1 = 0$. Show that H is *not* a subspace of $\mathbb{R}^2$.

5a(8 pts). I was applying the Gram-Schmidt process to the three vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ -3 \\ -1 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 3 \end{bmatrix}$$

and got as far as

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

when I was interrupted. Continue the Gram-Schmidt process to find a third vector $\mathbf{u}_3$.

5b(4 pts). What is the projection of $\mathbf{a}_3$ onto $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$?

5c(6 pts). Find the matrix R for which

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ -1 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} R$$

5d(6 pts). The vector $\mathbf{z} = \begin{bmatrix} -4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ is in the column space of A, since $\mathbf{z} = A \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

Find the coordinates of $\mathbf{z}$ relative to the orthonormal basis $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

6(26 pts). Let $E = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 5 \\ 0 & -1 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -7 \\ -20 \\ 2 \end{bmatrix}$.

a. Solve $E\mathbf{x} = \mathbf{b}$. State your answer (if it exists) in parametric vector form.

b. State the general solution to $E\mathbf{x} = \mathbf{0}$ in parametric vector form.

c. Give a basis for the null space of E.

d. What is the rank of E?

e. Is 0 an eigenvalue of E?

7(18 pts). Find Z^{-1} if $Z = \begin{bmatrix} -2 & -5 & -4 \\ 3 & 7 & 8 \\ 1 & 2 & 3 \end{bmatrix}$, or show that it does not exist.

8a(14 pts). Find all eigenvalues of $W = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

8b(4 pts). Are there any vectors $\mathbf{x} \neq \mathbf{0}$ satisfying $W\mathbf{x} = 5\mathbf{x}$? Why or why not?

9(14 pts). True or False?

T F a. If B is square, then $\det(B) = -\det(B^T)$.

T F b. If $\mathbf{x}$ is an eigenvector of G, then $\mathbf{x}$ is also an eigenvecor of G^2.

T F c. The dimensions of the column space of A and of the row space of A are not necessarily the same.

T F d. A transformation R from $\mathbb{R}^3$ into $\mathbb{R}^4$ is one-to-one if $\mathbf{u} = \mathbf{v}$ whenever $R(\mathbf{u}) = R(\mathbf{v})$.

T F e. If the sizes of the matrices G and H allow them to be multiplied, then the columns of GH are linear combinations of the columns of G.

T F f. $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

T F g. If C^T is invertible, then C is also invertible.

1a(18 pts). This is only one of many correct solutions. Expand along the bottom row and then perform the replacement $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_1$:

$$\begin{vmatrix} -1 & 8 & 2 & 4 \\ 0 & 3 & -4 & -4 \\ 1 & -1 & 5 & 5 \\ 0 & 1 & 0 & 0 \end{vmatrix} = +1 \begin{vmatrix} -1 & 2 & 4 \\ 0 & -4 & -4 \\ 1 & 5 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & 7 & 9 \end{vmatrix}$$

Factor our -4 from row 2, then $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 7\mathbf{r}_2$:

$$-4 \begin{vmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 7 & 9 \end{vmatrix} = -4 \begin{vmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -4 \cdot (-1) \cdot 1 \cdot 2 = 8.$$

1b(2 pts). F is invertible because $\det(F) \neq 0$.

2a(2 pts). A is 5×4 (so that $(5 \times 4)(4 \times 1) = (5 \times 1)$).

2b(5 pts). For every $\mathbf{u} \in \mathbb{R}^5$, there's a $\mathbf{x} \in \mathbb{R}^n$ for which $S(\mathbf{x}) = \mathbf{u}$. Equivalently, the range of S is all of $\mathbb{R}^5$, or the standard matrix for S has a pivot in every row.

2c(5 pts). The standard matrix A for T has only 4 columns, so it can't have a pivot in each of its 5 rows. Therefore T cannot map $\mathbb{R}^4$ onto $\mathbb{R}^5$.

3(6 pts). The 4×4 matrix $U = [\mathbf{u}_1\, \mathbf{u}_2\, \mathbf{u}_3\, \mathbf{u}_4]$ can't have a pivot in each of its four rows, since $U\mathbf{x} = [1 \quad 1 \quad 0 \quad 0]^T$ has no solutions. That implies that U has at most 3 pivot columns. Therefore the homogeneous equation $U\mathbf{x} = \mathbf{0}$ has a free variable, hence infinitely many solutions.

4(12 pts). Solve one or the other for full credit.

a. $S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $S(\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, but

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore S is not linear.

b. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in H because $0 \cdot 1 - 2 \cdot 0 = 0$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in H because $1 \cdot 2 - 2 \cdot 1 = 0$ but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not in H because $1 \cdot 3 - 2 \cdot 1 \neq 0$. Since H is not closed under addition, it is not a subspace.

5a(8 pts).

$$\begin{aligned}\mathbf{u}_3 &= \mathbf{a}_3 - \operatorname{proj}_{\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2\}} \mathbf{a}_3 \\ &= \mathbf{a}_3 - \left(\frac{\mathbf{u}_1\cdot\mathbf{a}_3}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}_2\cdot\mathbf{a}_3}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2\right) \\ &= \begin{bmatrix}3\\-5\\1\\3\end{bmatrix} - \left(\frac{-6}{4}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \frac{10}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-5\\1\\3\end{bmatrix} - \begin{bmatrix}1\\-4\\-1\\4\end{bmatrix} = \begin{bmatrix}2\\-1\\2\\-1\end{bmatrix}\end{aligned}$$

You can check this answer by checking that $\mathbf{u}_3\cdot\mathbf{u}_1 = \mathbf{u}_3\cdot\mathbf{u}_2 = 0$.

5b(4 pts). As seen above, the projection of $\mathbf{a}_3$ onto $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2\}$ is $\begin{bmatrix}1\\-4\\-1\\4\end{bmatrix}$.

5c(6 pts). Observe that the columns of $Q = \begin{bmatrix}1/2 & 1/2\\1/2 & -1/2\\-1/2 & -1/2\\-1/2 & 1/2\end{bmatrix}$ were obtained by normalizing the orthogonal vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, and that $Q^TQ = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. We can find R by solving: $A = QR \Longrightarrow Q^TA = Q^TQR = R$. Therefore

$$R = \begin{bmatrix}1/2 & 1/2 & -1/2 & -1/2\\1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}\begin{bmatrix}1 & 3\\1 & 1\\-1 & -3\\-1 & -1\end{bmatrix} = \begin{bmatrix}2 & 4\\0 & 2\end{bmatrix}.$$

(Note: we should expect R to be upper triangular since the columns of Q are obtained from the columns of A by Gram-Schmidt.)

5d(6 pts). Let's call the orthonormal basis $\mathcal{V}$ and the two vectors in it $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.
Solution one: the coordinates of $\mathbf{z}$ are $\dfrac{\mathbf{v}_1\cdot\mathbf{z}}{\mathbf{v}_1\cdot\mathbf{v}_1} = -4$ and $\dfrac{\mathbf{v}_2\cdot\mathbf{z}}{\mathbf{v}_2\cdot\mathbf{v}_2} = -4$. That is,

$$[\mathbf{z}]_{\mathcal{V}} = \begin{bmatrix}-4\\-4\end{bmatrix}.$$

Solution two: Multiply both sides of $Q[\mathbf{z}]_{\mathcal{V}} = \mathbf{z}$ on the left by Q^T to obtain

$$Q^TQ[\mathbf{z}]_{\mathcal{V}} = [\mathbf{z}]_{\mathcal{V}} = Q^T\mathbf{z} = \begin{bmatrix}1/2 & 1/2 & -1/2 & -1/2\\1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}\begin{bmatrix}-4\\0\\4\\0\end{bmatrix} = \begin{bmatrix}-4\\-4\end{bmatrix}.$$

Solution three: $\mathbf{z} = A\begin{bmatrix} 2 \\ -2 \end{bmatrix} = QR\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $[\mathbf{z}]_{\mathcal{V}}$ must be $R\begin{bmatrix} 2 \\ -2 \end{bmatrix}$:

$$[\mathbf{z}]_{\mathcal{V}} = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}.$$

6a.(26 pts). Augment and row-reduce:

row operation	**result**
initial augmented matrix	$\begin{matrix} 1 & 0 & 1 & -7 \\ 3 & 1 & 5 & -20 \\ 0 & -1 & -2 & 2 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1$	$\begin{matrix} 1 & 0 & 1 & -7 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 2 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2$	$\begin{matrix} 1 & 0 & 1 & -7 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{matrix}$

Because there's a pivot in the augmented column, $E\mathbf{x} = \mathbf{b}$ has no solution.

6b. The reduced row echelon form of E is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution to $E\mathbf{x} = \mathbf{0}$ is $x_1 = -x_3$, $x_2 = -2x_3$, x_3 free. That is, $\mathbf{x} = x_3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

6c. $\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is a basis for the null space of E. So is any of

$$\left\{\begin{bmatrix} -3 \\ 6 \\ 3 \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} 5 \\ -10 \\ -5 \end{bmatrix}\right\}, \ldots$$

6d. The rank of a matrix is the dimension of both its column space and its row space, that is, the number of its pivots. The rank of E is 2.

6e. Solution one: λ is an eigenvalue of the matrix E if and only if $E - \lambda I$ is singular. Since E is singular, 0 *is* one of its eigenvalues.

Solution two: 0 is an eigenvalue of E since $E\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0} = 0\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

7(18 pts). Augment Z with the identity and row-reduce:

Forward phase:

row operation	result
initial matrix	$\begin{matrix} -2 & -5 & -4 & 1 & 0 & 0 \\ 3 & 7 & 8 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{matrix}$
$\mathbf{r}_3 \rightleftarrows \mathbf{r}_1$	$\begin{matrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 3 & 7 & 8 & 0 & 1 & 0 \\ -2 & -5 & -4 & 1 & 0 & 0 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1$	$\begin{matrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -3 \\ 0 & -1 & 2 & 1 & 0 & 2 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2$	$\begin{matrix} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{matrix}$

end forward phase

Matrix is in **row echelon form**, and we can see that Z is invertible, since it has a pivot in every row and column.

Backward phase:

row operation	result
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 + \mathbf{r}_3$ $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 3\mathbf{r}_3$	$\begin{matrix} 1 & 2 & 0 & -3 & -3 & 4 \\ 0 & 1 & 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{matrix}$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 2\mathbf{r}_2$	$\begin{matrix} 1 & 0 & 0 & -5 & -7 & 12 \\ 0 & 1 & 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{matrix}$

end backward phase

Therefore $Z^{-1} = \begin{bmatrix} -5 & -7 & 12 \\ 1 & 2 & -4 \\ 1 & 1 & -1 \end{bmatrix}$. Let's check: https://sagecell.sagemath.org/?q=bksilb

8a(14 pts). The eigenvalues are the zero of the characteristic polynomial

$$|W - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

The eigenvalues are $\lambda = 1$ and $\lambda = 2$.

8b(4 pts). There are no nonzero vectors $\mathbf{x} \neq \mathbf{0}$ satisfying $W\mathbf{x} = 5\mathbf{x}$ because 5 is not an eigenvalue of W.

9(14 pts). The order of a-g was not the same on all exams.

i.(Source: 2.1.17) The columns of GH are linear combinations of the columns of G.

T The product of a matrix and a vector is a linear combination of that matrix's columns using the vector's elements as weights. Each column of GH is G times the corresponding column of H.

ii.(Source: 5.Supp.5) If $\mathbf{x}$ is an eigenvector of G, then $\mathbf{x}$ is also an eigenvecor of G^2.

T If $\mathbf{x}$ is nonzero and $G\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $GG\mathbf{x} = G\lambda\mathbf{x} = \lambda G\mathbf{x} = \lambda^2\mathbf{x}$.

iii.(Source: 4.5.22) The dimensions of the column space of A and of the row space of A are not necessarily the same.

F These are always the same. Both equal the number of pivots in the reduced row echelon form of A.

iv.(Source: 1.9.23) A transformation R from $\mathbb{R}^3$ into $\mathbb{R}^4$ is one-to-one if $\mathbf{u} = \mathbf{v}$ whenever $R(\mathbf{u}) = R(\mathbf{v})$.

T This is the definition of "one-to-one."

v.(Source: 3.Supp.9) $\det(B) = -\det(B^T)$.

F $\det B$ and $\det B^T$ are always the same.

vi.(Source: 2.3.19) If C^T is invertible, then C is also invertible.

T A matrix is invertible if and only if its transpose is invertible.

vii.(Source: 6.1.21,22,32) $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

F The equation is true for *any* $\mathbf{u}$ and $\mathbf{v}$.:

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\
&= 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2
\end{aligned}$$