No notes, books, electronic devices, or outside materials of any kind.
Read each problem carefully and simplify your answers.
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.
Solve or **find the solution** always means to find the general solution, if it exists.
You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(16 pts). Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, a basis for $\mathbb{R}^2$, and observe that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a. Find the coordinates $[\mathbf{u}]_{\mathcal{B}}$ of $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ relative to this basis.

b. Find the vector $\mathbf{v}$ if its coordinates relative to $\mathcal{B}$ are $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

2(25 pts). Suppose U and V are row-equivalent, where

$$U = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for each of the following vector spaces.

a. $\operatorname{Col} U$ b. $\operatorname{Col} V$ c. $\operatorname{Row} U$ d. $\operatorname{Row} V$ e. $\operatorname{Nul} V$

3. Let $C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

a(9 pts). Find the characteristic polynomial of C.

b(12 pts). Find a basis for the eigenspace for C corresponding to the eigenvalue $\lambda = 3$.

c(6 pts). -1 is another eigenvalue of C. Is C diagonalizable? Why or why not?

4(8 pts). Suppose $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ is a basis for the vector space W, and that $S : W \to W$ is a linear transformation, and that matrix of S relative to $\mathcal{D}$ is

$$[S]_{\mathcal{D}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

Find $S(2\mathbf{d}_2 - \mathbf{d}_3)$.

5(10 pts). Suppose A is a 5×8 matrix.

a. Fill in the blank: The dimension of $\operatorname{Nul} A$ must be at least ______.
"0" is a correct answer but will get you no credit. For full credit on this part, you must fill in the blank with the largest possible correct number.

b. If $\operatorname{Nul} A$ has dimension 6, what are the dimensions of $\operatorname{Col} A$ and of $\operatorname{Row} A$?

6(14 pts). Answer **one** of the following parts. Clearly indicate which part you're answering.

a. Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for the vector space V. Show that the coordinate map from V to $\mathbb{R}^3$ is **onto**; that is, show that for any $\mathbf{x} = [\, x_1 \quad x_2 \quad x_3 \,]^T$ in $\mathbb{R}^3$, there's a vector $\mathbf{v}$ in V for which $[\mathbf{v}]_{\mathcal{B}} = \mathbf{x}$.

b. Suppose $S = RTR^{-1}$, where S, R, and T are square matrices of the same size. Show that if $\mathbf{x}$ is an eigenvector of T with eigenvalue 4, then $R\mathbf{x}$ is an eigenvector of S, also with eigenvalue 4.

1a(8 pts).(Source: 4.4.5,6) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} [\mathbf{u}]_{\mathcal{B}} = \mathbf{u}$, so

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{u} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

1b(8 pts).(Source: 4.4.1,2)

$$\mathbf{v} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

2a(5 pts).(Source: 4.3.13) The pivot columns of U form a basis for $\operatorname{Col} U$: $\left\{ \begin{bmatrix} a \\ e \\ i \end{bmatrix}, \begin{bmatrix} b \\ f \\ j \end{bmatrix} \right\}$.

2b(4 pts). The pivot columns of V form a basis for $\operatorname{Col} V$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

2c(5 pts). and 2d(4 pts). $\operatorname{Row} U$ and $\operatorname{Row} V$ are the same vector space. The pivot rows of V form a basis: $\{[1 \quad 0 \quad 1 \quad 0], [0 \quad 1 \quad 0 \quad -1]\}$.
The example

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

illustrates that the first and second columns of V might not be a basis for $\operatorname{Col} U$, and that the first and second rows of U might fail to be a basis for $\operatorname{Row} U$.
2e(7 pts). $\operatorname{Nul} V$ is the solution set to $V\mathbf{x} = \mathbf{0}$. The free variables are x_3 and x_4, and, in parametric vector form,

$$\mathbf{x} = \begin{bmatrix} -x_3 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

These two linearly independent vectors span $\operatorname{Nul} V$, so they form a basis: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

3a(9 pts).(Source: 5.2.9-13) The characteristic polynomial of C is the determinant

$$|C - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$
$$= (3-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (3-\lambda)\big((1-\lambda)^2 - 4\big)$$

If you want, this can be factored as $(3-\lambda)(\lambda^2-2\lambda-3) = (3-\lambda)(\lambda-3)(\lambda+1)$.

4a(12 pts).(Source: 5.1.13-16) The eigenspace is the nullspace of

$$C - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The free variables are x_1 and x_3. Continuing as in 3e, we find $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis.

5a(6 pts).(Source: 5.3.12-16) $\lambda = 3$ has geometric multiplicity 2, and so $\lambda = -1$ must have geometric multiplicity 1. Consequently, $\mathbb{R}^3$ has a basis consisting of (3) eigenvalues of C, so that C *is* diagonalizable.

6(8 pts).(Source: 5.4.5-6) The matrix $[S]_{\mathcal{D}}$ must satisfy this diagram:

$$\begin{tikzcd} V \arrow[r, "S"] \arrow[d, "{[\ ]_{\mathcal{D}}}"'] & V \arrow[d, "{[\ ]_{\mathcal{D}}}"] \\ \mathbb{R}^3 \arrow[r, "{[S]_{\mathcal{D}}}"] & \mathbb{R}^3 \end{tikzcd}$$

That is, $[S(\mathbf{v})]_{\mathcal{D}} = [S]_{\mathcal{D}}[\mathbf{v}]_{\mathcal{D}}$ for each $\mathbf{v}$ in V. Let $\mathbf{v} = 2\mathbf{d}_2 - \mathbf{d}_3$. Then $[\mathbf{v}]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and

$$[S(\mathbf{v})]_{\mathcal{D}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}.$$

So $S(\mathbf{v}) = -3\mathbf{d}_1 + 4\mathbf{d}_2 - 5\mathbf{d}_3$.

7(10 pts).(Source: 4.5.36-39)

a(5 pts). The dimension of $\operatorname{Nul} A$ must be at least 3. Since A is 5×8, it has at most 5 pivot columns and therefore at least 3 nonpivot columns, and the dimension of $\operatorname{Nul} A$ equals the number of nonpivot columns.

b(5 pts). If A has 6 nonpivot columns, then it has 2 pivot columns. The dimensions of $\operatorname{Col} A$ and of $\operatorname{Row} A$ both equal the number of pivots, or 2.

8a(14 pts).(Source: 4.4.28) The vector $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3$ is in V, and its coordinates are $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. That is, $[x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3]_{\mathcal{B}} = \mathbf{x}$, and the proof is complete.

8b(14 pts).(Source: 5.4.25) For $\mathbf{x}$ to be an eigenvector for T for the eigenvalue 4 means that $\mathbf{x}$ is a nonzero vector satisfying

$$T\mathbf{x} = 4\mathbf{x}.$$

The vector $R\mathbf{x}$ is nonzero, since $R\mathbf{x} = \mathbf{0}$ would imply $\mathbf{x} = R^{-1}\mathbf{0} = \mathbf{0}$. All that remains is to show that $SR\mathbf{x} = 4R\mathbf{x}$:

$$SRx = (RTR^{-1})Rx = RT(R^{-1}R)x = RTI\mathbf{x} = RT\mathbf{x} = R4\mathbf{x} = 4R\mathbf{x},$$

completing the proof.