No notes, books, electronic devices, or outside materials of any kind.
Read each problem carefully and simplify your answers.
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.
Solve or **find the solution** always means to find the general solution, if it exists.
You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(6 pts). Solve $D\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ for $\mathbf{x}$ if $D^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$.

2(10 pts). Suppose $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}$.

Find the products, if they exist:
a. AB b. $B^T A$ c. $B^T A^T$

3(13 pts). Find E^{-1} if $E = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -3 & 0 \\ -2 & -6 & 1 \end{bmatrix}$.

4a(13 pts). Find $\det(U)$ if $U = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -3 & -1 & -1 \end{bmatrix}$.

4b(2 pts). Is the transformation $\mathbf{x} \mapsto U\mathbf{x}$ from $\mathbb{R}^4$ into $\mathbb{R}^4$ one-to-one?

5(6 pts). Find $\det(A)$ if A is a 2×2 matrix satisfying $A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix}$.

6(12 pts). Let $V = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}$. Find the row 2, column 3 entry in V^{-1}. Hint: $\det(V) = 2$.

7(14 pts). Determine whether the given set is a subspace. Justify your conclusions.
a. $M = \{a + bt \mid a + b = 1\}$, a subset of the vector space $\mathbb{P}_1$.
b. $N = \left\{ \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \,\middle|\, a, b \text{ real} \right\}$, a subset of the vector space $M_{2\times 2}$.

8(14 pts). True or False? Assume all unspecified matrices are square and of the same size.

T F a. $\begin{vmatrix} 0 & 0 & 0 & z \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 2 \\ w & 0 & 1 & 0 \end{vmatrix} = wxyz.$

T F b. $(A^T BA)^T = A^T B^T A$.

T F c. $\det(A)$ equals the product the diagonal elements of A.

T F d. If A and B are invertible, then so is AB, and $(AB)^{-1} = A^{-1}B^{-1}$.

T F e. If H is a subspace of a vector space V and $\mathbf{0}$ is the zero vector in V, then $\mathbf{0} \in H$.

T F f. If A is $n \times n$ and $\mathbf{b}$ is $n \times 1$, and $A\mathbf{x} = \mathbf{b}$ has exactly one solution, then the columns of A span $\mathbb{R}^n$.

T F g. For $\mathbf{x}$ to be in $\operatorname{Nul} A$ (that is, the null space of A) means $A\mathbf{x} = \mathbf{0}$.

9(10 pts). Answer **one** of the following. Clearly indicate which part you are answering.

a. If A is invertible, explain why $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

b. Suppose that R is $m \times n$, that S is $n \times p$, and that the columns of S are linearly dependent. Explain why the columns of RS are linear dependent.

1(6 pts).(Source: 2.2.7-9) $\mathbf{x} = D^{-1}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}11\\-3\end{bmatrix}$

2(10 pts).(Source: 2.1.1,2,5,6)

a. $AB = \begin{bmatrix}1 & 2 & -1\\2 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\-2 & 0\\1 & 1\end{bmatrix} =$

$$\begin{bmatrix}1\cdot 2+2\cdot(-2)-1\cdot 1 & 1\cdot 3+2\cdot 0-1\cdot 1\\2\cdot 2+1\cdot(-2)+1\cdot 1 & 2\cdot 3+1\cdot 0+1\cdot 1\end{bmatrix} = \begin{bmatrix}-3 & 2\\3 & 7\end{bmatrix}.$$

b. B^TA is not defined because B^T has 3 columns and A has 2 rows.

c. $B^TA^T = (AB)^T = \begin{bmatrix}-3 & 2\\3 & 7\end{bmatrix}^T = \begin{bmatrix}-3 & 3\\2 & 7\end{bmatrix}.$

3(13 pts). (Source: 2.2.41,42, 3.3.11-16) Augment E with the identity and row-reduce.

row operation	**result**
(beginning matrix)	$\begin{matrix}0 & 1 & 0 & 1 & 0 & 0\\-1 & -3 & 0 & 0 & 1 & 0\\-2 & -6 & 1 & 0 & 0 & 1\end{matrix}$
$\mathbf{r}_1 \rightleftarrows \mathbf{r}_2$ $\mathbf{r}_1 \leftarrow -\mathbf{r}_1$	$\begin{matrix}1 & 3 & 0 & 0 & -1 & 0\\0 & 1 & 0 & 1 & 0 & 0\\-2 & -6 & 1 & 0 & 0 & 1\end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_1$	$\begin{matrix}1 & 3 & 0 & 0 & -1 & 0\\0 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & -2 & 1\end{matrix}$
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 - 3\mathbf{r}_2$	$\begin{matrix}1 & 0 & 0 & -3 & -1 & 0\\0 & 1 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & -2 & 1\end{matrix}$

Therefore $E^{-1} = \begin{bmatrix}-3 & -1 & 0\\1 & 0 & 0\\0 & -2 & 1\end{bmatrix}.$

4a(13 pts).(Source: 3.1.9-12, 3.2.7-9) You can compute the determinant by using row elimination, cofactor expansion, or combination of the two. Here's a solution using cofactor expansion along the first column.

$$\begin{vmatrix}1 & 0 & 0 & 1\\0 & 2 & 1 & 0\\2 & 0 & 1 & 0\\0 & -3 & -1 & -1\end{vmatrix} = 1\begin{vmatrix}2 & 1 & 0\\0 & 1 & 0\\-3 & -1 & -1\end{vmatrix} + 2\begin{vmatrix}0 & 0 & 1\\2 & 1 & 0\\-3 & -1 & -1\end{vmatrix}$$

Now expand along the second row of the first 3×3 and first row of the second 3×3:

$$1\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} = -2 + 2(-2+3) = 0$$

4b(2 pts).(Source: 3.2.21,22, 2.3.37,38) U is not invertible because $\det(U) = 0$. By The Invertible Matrix Theorem (Section 2.3), the transformation $\mathbf{x} \mapsto U\mathbf{x}$ is not one-to-one.

5(6 pts).(Source: 3.2.39,43,45,46) You don't need to find A in order to find its determinant:

$$\begin{aligned} \left| A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right| = |A| \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} &= \begin{vmatrix} 0 & 1 \\ 1 & 8 \end{vmatrix} \\ = |A|(-2) = -1 \quad &\Longrightarrow \quad |A| = \frac{1}{2}. \end{aligned}$$

But, if you tried to find A, the solution is

$$\begin{aligned} A = \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \frac{1}{-2} = \begin{bmatrix} 3/2 & -1/2 \\ 10 & -3 \end{bmatrix}. \end{aligned}$$

6(12 pts).(Source: 3.3.11-16) Can use Cramer's rule. If $V\mathbf{x} =$ the third column of the identity matrix, then

$$x_2 = \frac{\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & -1 \end{vmatrix}}{\det(V)} = \frac{-\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}}{2} = \frac{1}{2}.$$

Or, can use the formula $V^{-1} = \operatorname{adj}(V)/\det(V)$:

$$\begin{aligned} (V^{-1})_{2,3} = \frac{1}{\det(V)} \operatorname{adj}(V)_{2,3} &= \frac{1}{2} \text{the row 3, column 2 cofactor of } V \\ &= \frac{-\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}}{2} = \frac{1}{2}. \end{aligned}$$

You don't need the full adjoint of V to find its row 2, column 3 entry, but for the record, the adjoint is

$$\operatorname{adj}(V) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -5 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

If you didn't use Cramer's rule, you could still find the answer by row reduction. You only need the third column of V^{-1}, so augment and reduce like so:

$$[\,V \quad e_3\,] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

So column 3 of $V^{-1} = [\,0 \quad 1/2 \quad -1/2\,]^T$, whose row 2 element is 1/2. See all the calculations at https://sagecell.sagemath.org/?q=jqzptb

7a(2 pts).(Source: 4.1.5,6) M is not a subspace. The zero function $0 + 0t \notin M$ because $0 + 0 \neq 1$.

7b(12 pts).(Source: 4.1.21) $N = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;\middle|\; a, b \text{ real} \right\}$ is the span of the two vectors $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By Theorem 1 (section 4.1) N is a subspace.
You could instead show that N contains $\mathbf{0}$, is closed under addition, and is closed under scalar multiplication. For this, it helpful to think of N as the set of all 2×2 matrices whose $2, 2$ entry is zero and whose $2, 1$ and $1, 2$ entries are equal.

8a(2 pts).(Source: 3.2.1,11-14) True:

$$\begin{vmatrix} 0 & 0 & 0 & z \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 2 \\ w & 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} w & 0 & 1 & 0 \\ 0 & 0 & y & 0 \\ 0 & x & 0 & 2 \\ 0 & 0 & 0 & z \end{vmatrix} = \begin{vmatrix} w & 0 & 1 & 0 \\ 0 & x & 0 & 2 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{vmatrix} = wxyz.$$

8b(2 pts).(Source: 2.1.22) True. Use the rule $(XY)^T = Y^T X^T$ to obtain

$$(A^T B A)^T = A^T B^T (A^T)^T = A^T B^T A.$$

8c(2 pts).(Source: 3.1.42) False. For instance, $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, but the product of its diagonal elements is 1.

8d(2 pts).(Source: 2.2.12,13) False. If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$, not $A^{-1}B^{-1}$.

8e(2 pts).(Source: 4.1.27) True. $\mathbf{0} \in H$ is part of the definition of a subspace.

8f(2 pts).(Source: 2.3.29) True. If $A\mathbf{x} = \mathbf{b}$ has exactly one solution, then A has a pivot in every column. Since A is square, to have a pivot in every column means A has a pivot in every row. Hence the columns of A span $\mathbb{R}^n$.

8g(2 pts).(Source: 4.2.25) True. $\operatorname{Nul} A$ is defined as the solution set of $A\mathbf{x} = \mathbf{0}$.

9a(10 pts).(Source: 3.2.37) Since A is invertible, $AA^{-1} = I$. Take determinants, and use the rule $\det(XY) = \det(X)\det(Y)$:

$$\det(A)\det(A^{-1}) = \det(I) = 1.$$

Divide by $\det(A)$:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

9b(10 pts).(Source: 2.1.26,27,30) The columns of a matrix are linearly dependent if and only if its nullspace contains a nonzero vector. Since the columns of S are linearly dependent, $S\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. But then $(RS)\mathbf{x} = R(S\mathbf{x}) = R\mathbf{0} = \mathbf{0}$, so the columns of RS are also linearly dependent.