No notes, books, electronic devices, or outside materials of any kind.
Read each problem carefully and simplify your answers.
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.
Solve or **find the solution** always means to find the general solution, if it exists.
You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(5 pts). Calculate the product, if it exists.

a. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} =$ b. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$

2(5 pts). Give a specific example of something in the "span" of the two vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ other than one of these vectors itself.

3(14 pts). Write the solution (if it exists) in parametric form.

$$\begin{aligned} x_2 - x_3 &= 0 \\ 3x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 4x_2 - 7x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

4(6 pts). Is one of the vectors in the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ -7 \\ -2 \end{bmatrix} \right\}$ is a linear combination of the other two? How do you know?

5(20 pts). Write the solution (if it exists) in parametric vector form:

a. $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 2 & 1 & 0 & -2 \\ 3 & -1 & -5 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 2 & 1 & 0 & -2 \\ 3 & -1 & -5 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

6(6 pts). Do the columns of $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 2 & 1 & 0 & -2 \\ 3 & -1 & -5 & -2 \end{bmatrix}$ span $\mathbb{R}^3$? Why or why not?

7(12 pts). Suppose S is the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ that first reflects the point $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ about the (horizontal) x_1 axis and then rotates the result $5\pi/6$ radians counterclockwise. Find the standard matrix of S.

8(8 pts). Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation. Find $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$ if

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

9. Suppose $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

a(4 pts). Fill in the blanks: A has ___ rows and ___ columns.

b(6 pts). Find another solution $\mathbf{x}$ to $A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

10(14 pts). Answer **one** of the following. Clearly indicate which part you are answering.

a. Suppose $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are vectors in $\mathbb{R}^3$. If $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\{\mathbf{a}_2, \mathbf{a}_3\}$ is linearly independent, must $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be linearly independent? Either explain why the answer is yes, or give a specific example to show that the answer is no.

b. Suppose P is an $m \times n$ matrix. If there is no vector $\mathbf{c}$ in $\mathbb{R}^m$ for which $P\mathbf{x} = \mathbf{c}$ has infinitely many solutions, how do m and n compare? (That is, what can you conclude about their order on the real number line?) How do you know?

1(5 pts).(Source: 1.4.1-4) a. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$.

b. Does not exist. The matrix has more columns than the vector has rows.

2(5 pts).(Source: 1.3.15-16) The span consists of all linear combinations $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For instance, $2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ is in the span.

3(14 pts).(Source: 1.1.11-14,19-20, 1.2.7-14, 1.3.13-14)
Write this homogeneous system in augmented form and row reduce.

Forward phase:

row operation	**result**
initial matrix	$\begin{array}{cccc} 0 & 1 & -1 & 0 \\ 3 & 4 & -6 & 0 \\ 4 & 4 & -7 & 0 \\ 1 & 1 & -2 & 0 \end{array}$
$\mathbf{r}_4 \rightleftarrows \mathbf{r}_1$	$\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 3 & 4 & -6 & 0 \\ 4 & 4 & -7 & 0 \\ 0 & 1 & -1 & 0 \end{array}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 4\mathbf{r}_1$	$\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}$
$\mathbf{r}_4 \leftarrow \mathbf{r}_4 - \mathbf{r}_2$	$\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}$
$\mathbf{r}_4 \leftarrow \mathbf{r}_4 + \mathbf{r}_3$	$\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$

end forward phase

Matrix is in **row echelon form**, and we can identify the pivots. There's a pivot in every column of the coefficient matrix, so the only solution to the homogeneous system is the trivial solution $\mathbf{x} = \mathbf{0}$. That is, $x_1 = x_2 = x_3 = 0$.

4(6 pts).(Source: 1.7.5-8,39-42) **No.** See Theorem 7, section 1.7. This question asks if the vectors in question are linearly dependent. In 3, we found that there was a pivot in each column of the coefficient matrix, so its columns (seen again in 4) are linearly independent.

5.(Source: 1.5-6&19-20, 1.2.7-14, 1.3.13-14)

a(16 pts). Write the system in augmented form and row reduce.

Forward phase:

row operation	**result**
initial matrix	$\begin{matrix} 1 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & -2 & -3 \\ 3 & -1 & -5 & -2 & 1 \end{matrix}$
$\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1$ $\mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_1$	$\begin{matrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & -1 & -2 & 1 & 4 \end{matrix}$
$\mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2$	$\begin{matrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{matrix}$

end forward phase

Matrix is in **row echelon form**. Can identify the pivots, but not the solution.

Backward phase:

row operation	**result**
$\mathbf{r}_1 \leftarrow \mathbf{r}_1 + \mathbf{r}_3$	$\begin{matrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{matrix}$

end backward phase

Matrix is in **reduced row echelon form**.

Solution in ... parametric form: parametric vector form (required):

$$\begin{aligned} x_1 &= 2 + x_3 \\ x_2 &= -1 - 2x_3 \\ x_3 &= \text{ free} \\ x_4 &= 3 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

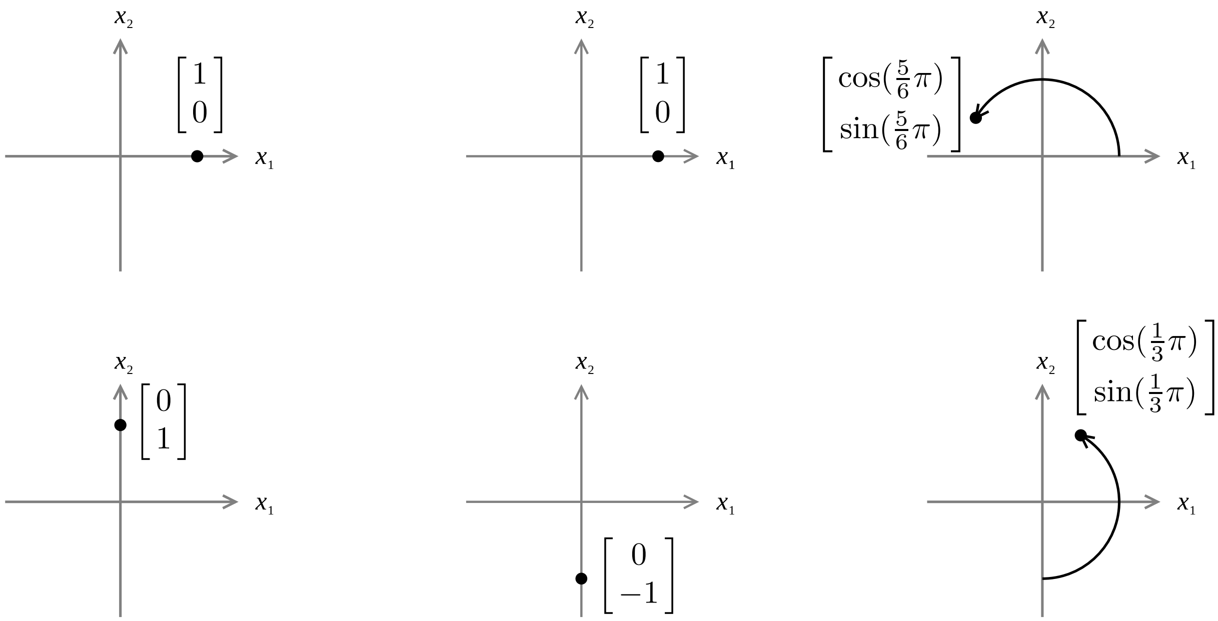
b(4 pts). In the homogeneous problem, augmented column would be $\mathbf{0}$ throughout, so solution is $\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. To see it another way, use Theorem 6, section 1.5. Since $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$ is a solution of the nonhomogeneous system, we can subtract it from the general solution of the non-homogeneous system to obtain the general solution of the homogeneous system.

6(6 pts).(Source: 1.4.17-22) **Yes**. See Theorem 4, section 1.4. Matrix has a pivot in every row, so its columns span $\mathbb{R}^3$.

7(12 pts).(Source: 1.9.10) See problem 1.9.re2 in the review notes, found at
https://kunklet.people.charleston.edu/MATH203/203review.pdf

To find the columns of its standard matrix, find the image under S of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$:

Starting point | reflected about $x_2 = 0$ | then rotated $\frac{5\pi}{6}$ radians:



Therefore the standard matrix is $A = \begin{bmatrix}\cos(\frac{5}{6}\pi) & \cos(\frac{1}{3}\pi)\\ \sin(\frac{5}{6}\pi) & \sin(\frac{1}{3}\pi)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\sqrt{3} & 1\\ 1 & \sqrt{3}\end{bmatrix}$.

8(8 pts).(Source: 1.8.19, 1.9.1-2) $T : \mathbb{R}^2 \to \mathbb{R}^3$ is **linear** means

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \in \mathbb{R}^2$$

and

$$T(c\mathbf{u}) = cT(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{R}^2 \text{ and } c \in \mathbb{R}$$

So,

$$\begin{aligned} T(\begin{bmatrix}2\\3\end{bmatrix}) &= T(2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}) = T(2\begin{bmatrix}1\\0\end{bmatrix}) + T(3\begin{bmatrix}0\\1\end{bmatrix}) = 2T(\begin{bmatrix}1\\0\end{bmatrix}) + 3T(\begin{bmatrix}0\\1\end{bmatrix}) \\ &= 2\begin{bmatrix}2\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}7\\8\\2\end{bmatrix} \end{aligned}$$

In fact, $\begin{bmatrix}2 & 1\\1 & 2\\1 & 0\end{bmatrix}$ is the standard matrix for T, and $T(\begin{bmatrix}2\\3\end{bmatrix}) = \begin{bmatrix}2 & 1\\1 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}7\\8\\2\end{bmatrix}$.

9a(4 pts).(Source: 1.4.5-6) A has 2 rows and 3 columns. See problem 1a, for example.

9b(6 pts).(Source: 1.4.45,46) $\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}2\\-1\\2\end{bmatrix}$ is a solution because $A\left(\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-2\\1\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$. For that matter, $\begin{bmatrix}1\\1\\1\end{bmatrix}+s\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is a solution for any scalar s. **(done)**

There are infinitely many such matrices A, so finding one and solving $A\mathbf{x}=\begin{bmatrix}2\\3\end{bmatrix}$ doesn't help. For instance, the solutions when $A=\begin{bmatrix}2/3&2/3&2/3\\1&1&1\end{bmatrix}$ are not the same as the solutions when $A=\begin{bmatrix}4/3&2/3&0\\0&1&2\end{bmatrix}$. But $A\begin{bmatrix}2\\-1\\2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$ is true for *any* such A.

10(14 pts).

a.(Source: 1.7.26-27) **No.** $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ is linearly independent, since there's a pivot in each column. $\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ is linearly independent, since it's the same set. But $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ is linearly dependent, since the first and the third columns are the same. Here's a nontrivial way to obtain $\mathbf{0}$ as a linear combination of these:

$$\begin{bmatrix}1\\0\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

b.(Source: 1.4.43, 1.7.25) In any $m\times n$ matrix,

$$\text{number of pivots} \le m \quad \text{and} \quad \text{number of pivots} \le n,$$

because no column or row can contain two pivots.

When $\mathbf{c}=\mathbf{0}$, the statement of the problem says that $P\mathbf{x}=\mathbf{0}$ has infinitely many solutions, so P has a pivot in every column. Therefore the number of pivots = the number of columns = $n \le m$ = the number of rows.