0.1: Real numbers and sets

A **set** is any collection of objects. Even a collection of *no* objects is a set, called the **empty set** and written $\emptyset$. Many questions of calculus involve sets of numbers.
The symbol $\mathbb{R}$ stands for the set of all **real numbers**, and the symbol $\mathbb{Z}$ stands for the set of all **integers** (whole real numbers). The symbol $\mathbb{N}$ is sometimes used for the set of positive integers, or **natural numbers**.

Two ways to write a set in set notation

1. By listing all its elements.

0.1.r1. $A = \{0, 1, \sqrt{2}, 2\}$ is the set of the four numbers 0, 1, 2, and $\sqrt{2}$.

The symbol $\in$ means "is an element of (the set)," or "belongs to (the set)," and $\notin$ means " is not an element of."

0.1.r1, continued. $\sqrt{2} \in A$, but $3 \notin A$.

0.1.r2. The empty set $\emptyset$ can also be written $\{\}$.

0.1.r3. $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, or $\{0, 1, 2, 3, \ldots, 11\}$. The three dots mean "and so on," and can be used after listing enough terms to indicate the pattern.

0.1.r4. $\Gamma = \{1, 3, 5, \ldots, 99\}$ is the set of odd integers from 1 to 99. $\mathcal{F} = \{1, 3, 5, \ldots\}$ is the set of all positive odd integers. $\mathcal{E} = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$ is the set of all even integers.

2. By describing all its elements using a dummy variable.

0.1.r5. The set of even integers can be written $\{2n \mid n \in \mathbb{Z}\}$. You could write the same set using a different name for the dummy variable, e.g., $\{2m \mid m \in \mathbb{Z}\}$ or $\{2\beta \mid \beta \in \mathbb{Z}\}$.

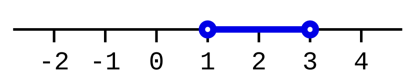
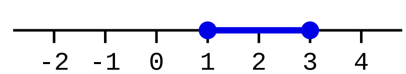
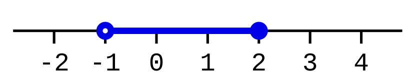
0.1.r6. $\{x \in \mathbb{R} \mid x^2 = 9\}$ is the set of all real numbers whose square equals 9, or $\{-3, 3\}$.

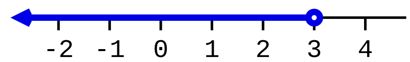
0.1.r7. $\{\alpha \in \mathbb{R} \mid 1 < \alpha < 3\}$ is the set of numbers strictly between 1 and 3.

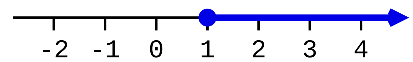
Intervals

The set in Example 0.1.r7 is an **interval**, a set whose elements form a segment on the real number line. An interval may or may not include its endpoints and can have infinite length. **Interval notation** is a convenient way to denote an interval.

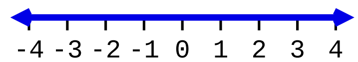
0.1.r8. Different ways to represent an interval.

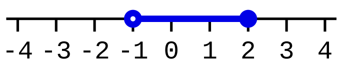
interval notation	**set notation**	**graph**
$(1, 3)$	$\{s \in \mathbb{R} \mid 1 < s < 3\}$	
$[1, 3]$	$\{t \in \mathbb{R} \mid 1 \leq t \leq 3\}$	
$(-1, 2]$	$\{u \in \mathbb{R} \mid -1 < u \leq 2\}$	

$(-\infty, 3)$ $\{w \in \mathbb{R} \mid w < 3\}$ 

$[1, \infty)$ $\{x \in \mathbb{R} \mid 1 \leq x\}$ 

0.1.r9. Find the missing interval notation, set notation, or graph of each interval.

a. $[-2, 2)$
b. $\{s \in \mathbb{R} \mid -3 < s < 2\}$
c. 

d. $\{x \in \mathbb{R} \mid x > -2\}$
e. $[0, 3.5]$
f. 

g. $\{w \in \mathbb{R} \mid w \leq 3\}$

Absolute value

The **absolute value** of a real number is its distsnce from the origin on the number line and is given by this piecewise-algebraic formula:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \text{ and} \\ -x & \text{if } x < 0. \end{cases}$$

$|x| \geq 0$ for all real numbers x.

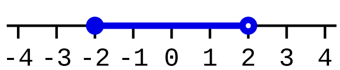
0.1.r10. 5 and 0 are both ≥ 0, so their absolute values are $|5| = 5$ and $|0| = 0$. The number $-3 < 0$, so its absolute value is $|-3| = -(-3) = 3$.

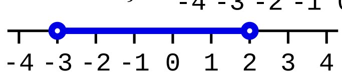
0.1.r11. Find the two numbers with absolute value 5.

0.1.r12. Find the solutions to the given equation.

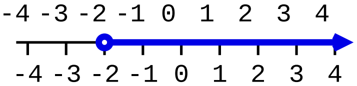
a. $|x| = 3.5$ b. $|u| = 0$ c. $|y| = -2$

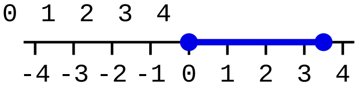
Answers

0.1.r9a. $\{v \in \mathbb{R} \mid -2 \leq v < 2\}$ 

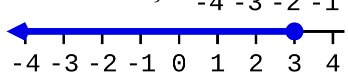
0.1.r9b. $(-3, 2)$ 

0.1.r9c. $(-\infty, \infty)$, $\mathbb{R}$

0.1.r9d. $(-2, \infty)$ 

0.1.r9e. $\{t \in \mathbb{R} \mid 0 \leq t \leq 3.5\}$ 

0.1.r9f. $(-1, 2]$, $\{u \in \mathbb{R} \mid -1 < u \leq 2\}$.

0.1.r9g. $(-\infty, 3]$ 

0.1.r11. 5 and -5 0.1.r12a. $x = 3.5$ or -3.5.
0.1.r12b. $u = 0$ is the only solution. 0.1.r12c. No solutions. There is no number whose absolute value is negative.

0.1a: Arithmetic with fractions

To add fractions with the same denominator, add the numerators:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \tag{1}$$

To multiply fractions, multiply the numerators and denominators:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \tag{2}$$

To divide fractions, invert and multiply:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \tag{3}$$

Division by zero is not allowed:

$$\frac{a}{0}\text{ does not exist} \tag{4}$$

because $\frac{a}{0}=?$ would mean $a=0\cdot ?$, which has no solution if $a\neq 0$ and infinitely many solutions if $a=0$.
As a consequence of (2),

$$\frac{ac}{bc}=\frac{a}{b} \tag{5}$$

(because $\frac{ac}{bc}=\frac{a}{b}\cdot\frac{c}{c}=\frac{a}{b}\cdot 1$.) We use this either to simplify a fraction by canceling common factors, e.g.,

$$\frac{8}{12}=\frac{2\cdot 4}{3\cdot 4}=\frac{2}{3}$$

or to obtain common denominators for adding fractions, e.g.,

$$\frac{4}{3}+\frac{3}{5}=\frac{5\cdot 4}{5\cdot 3}+\frac{3\cdot 3}{5\cdot 3}=\frac{20}{15}+\frac{9}{15}=\frac{29}{15}.$$

There's nothing inherently wrong with an improper fraction, but to write an improper fraction as a mixed number, we use long division. For example,

$$17\div 5=3\text{ with remainder }2$$

means that

$$\frac{17}{5}=3+\frac{2}{5}.$$

0.1a.r1. Find the sum or difference.

a. $\frac{2}{5}+\frac{4}{5}$ b. $\frac{8}{13}-\frac{5}{13}$ c. $\frac{2}{9}+\frac{8}{9}$ d. $\frac{2}{8}-\frac{3}{8}$ e. $\frac{7}{6}+\frac{5}{6}$ f. $\frac{4}{3}-\frac{7}{3}$

g. $\frac{2}{3}+\frac{4}{5}$ h. $\frac{3}{4}-\frac{1}{5}$ i. $\frac{13}{3}+\frac{1}{8}$ j. $\frac{2}{11}+\frac{6}{7}$ k. $\frac{4}{21}-\frac{3}{11}$ l. $\frac{4}{5}+\frac{5}{4}$

m. $\frac{5}{14}-\frac{1}{15}$ n. $\frac{6}{52}+\frac{4}{3}$ o. $\frac{5}{12}-\frac{3}{5}$ p. $\frac{2}{3}+\frac{4}{7}$ q. $\frac{7}{10}-\frac{4}{9}$ r. $\frac{1}{30}-\frac{11}{7}$

0.1a.r2. Find the sum or difference. Hint: always use the *least* common demonimator.

a. $\frac{3}{4}-\frac{3}{10}$ b. $\frac{2}{9}+\frac{4}{21}$ c. $\frac{1}{22}+\frac{3}{8}$ d. $\frac{3}{10}-\frac{2}{15}$ e. $\frac{7}{8}+\frac{17}{12}$ f. $\frac{5}{18}-\frac{5}{24}$

g. $\frac{6}{10}-\frac{7}{15}$ h. $\frac{2}{5}+\frac{4}{25}$ i. $\frac{8}{3}+\frac{7}{9}$ j. $\frac{5}{8}-\frac{5}{24}$ k. $\frac{7}{11}+\frac{6}{55}$ l. $\frac{5}{9}+\frac{2}{15}$

m. $\frac{25}{21}-\frac{17}{15}$ n. $\frac{25}{24}+\frac{17}{20}$ o. $\frac{18}{15}-\frac{40}{35}$ p. $\frac{13}{6}+\frac{15}{8}$ q. $\frac{25}{12}-\frac{9}{8}$ r. $\frac{31}{10}+\frac{21}{16}$

0.1a.r3. Find the sum or difference. Hint: $n=\frac{n}{1}$

a. $\frac{2}{3}-4$ b. $\frac{4}{9}+2$ c. $\frac{13}{4}-2$ d. $\frac{11}{14}+3$ e. $\frac{7}{4}-5$ f. $\frac{8}{13}+2$

g. $10-\frac{5}{7}$ h. $5+\frac{4}{5}$ i. $4-\frac{1}{3}$ j. $3+\frac{2}{13}$ k. $2-\frac{5}{4}$ l. $-3+\frac{4}{9}$

0.1a.r4. Find the product. Hint: cancel common factors before multiplying.

a. $\frac{2}{3}\times\frac{0}{4}$ b. $\frac{5}{7}\times\frac{4}{3}$ c. $\frac{5}{0}\times\frac{11}{6}$ d. $\frac{2}{15}\times\frac{3}{5}$ e. $\frac{1}{3}\times\frac{25}{7}$ f. $\frac{-8}{5}\times\frac{2}{11}$

g. $\frac{8}{15}\times\frac{3}{10}$ h. $\frac{3}{-2}\times\frac{5}{6}$ i. $\frac{-3}{2}\times\frac{-7}{5}$ j. $\frac{-9}{-4}\times\frac{6}{6}$ k. $\frac{5}{4}\times\frac{6}{7}$ l. $\frac{14}{5}\times\frac{45}{21}$

m. $\frac{9}{2}\times 3$ n. $3\times\frac{4}{5}$ o. $\frac{3}{4}\times 12$ p. $\frac{2}{9}\times(-6)$ q. $4\times(-\frac{4}{11})$ r. $\frac{4}{16}\times 24$

0.1a.r5. Find the quotient. Remember that division by zero is not allowed.

a. $\frac{2}{5}\div\frac{11}{2}$ b. $\frac{8}{5}\div\frac{12}{35}$ c. $\frac{1}{2}\div\frac{-5}{7}$ d. $\frac{3}{14}\div\frac{12}{7}$ e. $\frac{1}{8}\div\frac{1}{9}$ f. $\frac{0}{3}\div\frac{1}{2}$

g. $\frac{8}{11}\div\frac{6}{5}$ h. $\frac{3}{-14}\div\frac{2}{-21}$ i. $\frac{3}{0}\div\frac{5}{1}$ j. $\frac{3}{7}\div\frac{9}{2}$ k. $\frac{4}{0}\div\frac{4}{9}$ l. $\frac{14}{5}\div\frac{7}{13}$

m. $\frac{1}{3}\div 4$ n. $\frac{2}{5}\div 2$ o. $\frac{5}{3}\div 3$ p. $\frac{2}{7}\div 8$ q. $\frac{-2}{7}\div 3$ r. $\frac{5}{3}\div 8$

Answers

0.1a.r1a. $\frac{6}{5}$ 0.1a.r1b. $\frac{3}{13}$ 0.1a.r1c. $\frac{10}{9}$ 0.1a.r1d. $-\frac{1}{8}$ 0.1a.r1e. 2 0.1a.r1f. -1 0.1a.r1g. $\frac{22}{15}$ 0.1a.r1h. $\frac{11}{20}$ 0.1a.r1i. $\frac{107}{24}$ 0.1a.r1j. $\frac{80}{77}$ 0.1a.r1k. $-\frac{19}{231}$ 0.1a.r1l. $\frac{41}{20}$ 0.1a.r1m. $\frac{61}{210}$ 0.1a.r1n. $\frac{113}{78}$ 0.1a.r1o. $-\frac{11}{60}$ 0.1a.r1p. $\frac{26}{21}$ 0.1a.r1q. $\frac{23}{90}$ 0.1a.r1r. $-\frac{323}{210}$ 0.1a.r2a. $\frac{9}{20}$ 0.1a.r2b. $\frac{26}{63}$ 0.1a.r2c. $\frac{37}{88}$ 0.1a.r2d. $\frac{1}{6}$ 0.1a.r2e. $\frac{55}{24}$ 0.1a.r2f. $\frac{5}{72}$ 0.1a.r2g. $\frac{2}{15}$ 0.1a.r2h. $\frac{14}{25}$ 0.1a.r2i. $\frac{31}{9}$ 0.1a.r2j. $\frac{10}{24}$ 0.1a.r2k. $\frac{41}{55}$ 0.1a.r2l. $\frac{31}{45}$ 0.1a.r2m. $\frac{2}{35}$ 0.1a.r2n. $\frac{23}{120}$ 0.1a.r2o. $\frac{2}{35}$ 0.1a.r2p. $\frac{107}{24}$ 0.1a.r2q. $\frac{23}{24}$ 0.1a.r2r. $\frac{353}{80}$ 0.1a.r3a. $-\frac{10}{3}$ 0.1a.r3b. $\frac{22}{9}$ 0.1a.r3c. $\frac{5}{4}$ 0.1a.r3d. $\frac{53}{14}$ 0.1a.r3e. $-\frac{13}{4}$ 0.1a.r3f. $\frac{34}{13}$ 0.1a.r3g. $\frac{65}{7}$ 0.1a.r3h. $\frac{29}{5}$ 0.1a.r3i. $\frac{11}{3}$ 0.1a.r3j. $\frac{41}{13}$ 0.1a.r3k. $\frac{3}{4}$ 0.1a.r3l. $-\frac{23}{9}$ 0.1a.r4a. 0 0.1a.r4b. $\frac{20}{21}$ 0.1a.r4c. DNE 0.1a.r4d. $\frac{2}{25}$ 0.1a.r4e. $\frac{25}{21}$ 0.1a.r4f. $-\frac{16}{55}$ 0.1a.r4g. $\frac{4}{25}$ 0.1a.r4h. $-\frac{5}{4}$ 0.1a.r4i. $\frac{21}{10}$ 0.1a.r4j. $\frac{9}{4}$ 0.1a.r4k. $\frac{15}{14}$ 0.1a.r4l. 6 0.1a.r4m. $\frac{27}{2}$ 0.1a.r4n. $\frac{12}{5}$ 0.1a.r4o. 9 0.1a.r4p. $-\frac{4}{3}$ 0.1a.r4q. $-\frac{16}{11}$ 0.1a.r4r. 6 0.1a.r5a. $\frac{4}{55}$ 0.1a.r5b. $\frac{14}{3}$ 0.1a.r5c. $-\frac{7}{10}$ 0.1a.r5d. $\frac{1}{8}$ 0.1a.r5e. $\frac{9}{8}$ 0.1a.r5f. 0 0.1a.r5g. $\frac{520}{121}$ 0.1a.r5h. $\frac{9}{4}$ 0.1a.r5i. DNE 0.1a.r5j. $\frac{2}{21}$ 0.1a.r5k. DNE 0.1a.r5l. $\frac{26}{5}$ 0.1a.r5m. $\frac{1}{12}$ 0.1a.r5n. $\frac{1}{5}$ 0.1a.r5o. $\frac{5}{9}$ 0.1a.r5p. $\frac{2}{56}$ 0.1a.r5q. $-\frac{2}{21}$ 0.1a.r5r. $\frac{5}{24}$

0.2: Integer exponents

Exponents denote repeated multiplication or division e.g.,

$$a^3 = a \cdot a \cdot a \qquad a^{-4} = \frac{1}{a \cdot a \cdot a \cdot a} \qquad a^0 = 1$$

Laws of exponents 0.2.r1.

$$a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad \frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}.$$

It's easiest to remember these laws by thinking of simple examples.

0.2.r2. $a^3a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) = a^5$ (not a^6)

0.2.r3. $(a^3)^2 = (a \cdot a \cdot a)^2 = (a \cdot a \cdot a) \cdot (a \cdot a \cdot a) = a^6$ (not a^5)

0.2.r4. $(ab)^3 = (ab)(ab)(ab) = a \cdot a \cdot a \cdot b \cdot b \cdot b = a^3b^3$

These laws allow us to rewrite a quantity that depends on two operations according to which operation we want to do first. For instance $(ab)^n = a^nb^n$ says that we can multiply a and b and raise the result to the power n, or we can raise both a and b to the power n and multiply the results, and the outcome will be the same either way.
That's remarkable because, generally, order of operations *does* matter. For instance,

$$(4+3)^2 = 7^2 = 49 \quad \text{but} \quad 4^2 + 3^2 = 16 + 9 = 25.$$

Since order of operations matters, algebra follows the rule that operations are performed in the order they are written left to right, except that

Operations in
Parentheses
are executed before
Exponents,
which are executed before
Multiplication and Division,
which are executed before
Addition and Subtraction.

or PEMDAS for short.

0.2.r5. As an exercise, evaluate the following expressions without a caluclutor.

a. $4 + 3 \cdot 2 = 4 + 6 = 10$
b. $(4+3) \cdot 2 = 7 \cdot 2 = 14$
c. $-2^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$
d. $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

0.2.r6. A long fraction bar tells us to evaluate the top and bottom before performing the division, just as if there were parentheses around the numerator and around the denominator, e.g.,

$$\frac{7+5}{1+2\cdot 7} = \frac{(7+5)}{(1+2\cdot 7)} = \frac{12}{1+14} = \frac{12}{15} = \frac{3\cdot 4}{3\cdot 5} = \frac{4}{5}$$

0.2.r7. Evaluate the followiing expressions without a calculator.

a. $-3\cdot 6+4\cdot 3$

b. $2\cdot(3-5)+8\cdot 2-1$

c. $1-(4\cdot 3-2+5)$

d. $10-\left(6-2\cdot 2+(8-3)^2\right)\cdot 2$

e. $1-(3\cdot 3^3-2+5)$

f. $1-\dfrac{1}{4\cdot 3-2}$

g. $\dfrac{2^3+1}{3\cdot 2^2-5\cdot 6+15}$

0.2.r8. Rewrite without parentheses. Express your answer without negative exponents, and again without fractions. Assume all variables are nonzero.

a. $(x^3y^{-5})^2$

b. $(u^{-2}v^3)^{-6}$

c. $y^{-5}(x^2y^2x^{-4})^3$

d. $a^3bb^{-1}a^{-4}$

e. $(xy^2(x^2y^{-2})^3yx^5)^{-1}$

f. $a^{-3}b^4(ab^3)^{-2}$

Answers

0.2.r7a. -6 0.2.r7b. 11 0.2.r7c. -14 0.2.r7d. 56 0.2.r7e. -83 0.2.r7f. $\frac{9}{10}$ 0.2.r7g. -3
0.2.r8a. $\frac{x^6}{y^{10}} = x^6y^{-10}$ 0.2.r8b. $\frac{u^{12}}{v^{18}} = u^{12}v^{-18}$ 0.2.r8c. $\frac{y}{x^6} = x^{-6}y$ 0.2.r8d. $a^{-1} = \frac{1}{a}$ 0.2.r8e. $\frac{y^3}{x^{12}} = x^{-12}y^3$ 0.2.r8f. $\frac{1}{b^2a^5} = a^{-5}b^{-2}$

0.3: Fractional exponents and radicals

$\sqrt[n]{a}$ refers to a solution to the equation $x^n = a$, if one exists.
If n is odd, then every real number has exactly one nth root.

0.3.r1. $\sqrt[3]{-8} = -2$, since -2 is the unique solution to the equation $x^3 = -8$.

If n is even, then negative numbers don't posses real nth roots, and every positive number has two real nth roots, one positive and one negative. In that case, the symbol $\sqrt[n]{a}$ stands for the **positive** nth root of a. The radical symbol $\sqrt{}$ with no index is always assumed to mean the square root $\sqrt[2]{}$.

0.3.r2. Both 3 and -3 are solutions to $x^2 = 9$, so both 3 and -3 are square roots of 9. The symbol $\sqrt{9}$ stands for the *positive* square root, or 3.

0.3.r3. $\sqrt{-9}$ does not exist, since $x^2 = -9$ has no real solutions.

Arithmetic tip: to evaluate a radical, it helps to factor the radicand (the number under the radical) and use

$$\sqrt[n]{x}\,\sqrt[n]{y} = \sqrt[n]{xy}$$

0.3.r4. $1600 = 16 \cdot 100$, so $\sqrt{1600} = \sqrt{16}\sqrt{100} = 4 \cdot 10 = 40$.

0.3.r5. $216 = 2 \cdot 108 = 2^2 \cdot 54 = 2^3 \cdot 27 = \cdots 2^3 \cdot 3^3$, so

a. $\sqrt[3]{216} = \sqrt[3]{2^3}\,\sqrt[3]{3^3} = 2 \cdot 3 = 6$ b. $\sqrt[3]{-216} = \sqrt[3]{-1}\,\sqrt[3]{216} = (-1)6 = -6$

We write the nth root of a as a power

$$a^{1/n} = \sqrt[n]{a},$$

since the laws of exponents tell us that $a^{1/n}$ must be a number whose nth power is a:

$$\left(a^{\frac{1}{n}}\right)^n = (a^{\frac{1}{n}\cdot n}) = a^{n/n} = a^1 = a.$$

Since $a^{\frac{m}{n}}$ can be viewed as either $\left(a^{\frac{1}{n}}\right)^m$ or $(a^m)^{\frac{1}{n}}$, we say

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

provided the root exists.

0.3.r6. $\sqrt[3]{x^5} = x^{5/3}$.

0.3.r7. $\frac{1}{\sqrt[3]{x^5}} = \frac{1}{x^{5/3}} = x^{-5/3}$.

0.3.r8. $y^{-4/3}$ can be written with radicals several ways:

$$(\sqrt[3]{y})^{-4} = \frac{1}{(\sqrt[3]{y})^4} = \frac{1}{\sqrt[3]{y^4}} = \sqrt[3]{\frac{1}{y^4}} = \sqrt[3]{y^{-4}}$$

0.3.r9. Rewrite the given roots as fractional powers, and vice versa. There may be multiple correct answers.

a. $a^{1/7}$ b. $\sqrt[4]{x^6}$ c. $b^{-9/4}$

d. $\sqrt[5]{\dfrac{1}{x^7}}$ e. $z^{5/2}$ f. $\dfrac{1}{\sqrt{x^3}}$

0.3.r10. Rewrite without parentheses. Express your answer without negative exponents, and again without fractions. Assume all variables are nonzero.

a. $(u^{-1/2}v^{1/3})^{-12}$ b. $(a^3bc^{-2}b^{-1}a^{-4})^{-1/2}$ c. $(uv^{-2}u^{-3}v^8)^{3/2}$

d. $(x(x^2y^{-2})^3yx^5)^{-1/3}$ e. $a^{-3}(ab^3)^{1/6}b^4$ f. $\sqrt{xy}(x^2y)^3(y^{-1}z)^4$

Answers

0.3.r9a. $\sqrt[7]{a}$ 0.3.r9b. $x^{6/4}$, or, more simply, $x^{3/2}$ 0.3.r9c. $(\sqrt[4]{y})^{-9}4$, or $\frac{1}{(\sqrt[4]{y})^9}$, or $\frac{1}{\sqrt[4]{y^9}}$, or ...
0.3.r9d. $x^{-7/5}$ 0.3.r9e. $\sqrt{x^5}$ 0.3.r9f. $x^{-3/2}$ 0.3.r10a. $\frac{u^6}{v^4} = u^6v^{-4}$ 0.3.r10b. $a^{1/2}c$ 0.3.r10c. $\frac{v^9}{u^3} = u^{-3}v^9$
0.3.r10d. $\frac{y^{5/3}}{x^4} = x^{-4}y^{5/3}$ 0.3.r10e. $\frac{b^{9/2}}{a^{-17/6}} = a^{17/6}b^{9/2}$ 0.3.r10f. $\frac{x^{13/2}z^4}{y^{1/2}} = x^{13/2}y^{-1/2}z^4$

0.4: Polynomials

A **monomial** is real number (called its **coefficient**) times a nonnegative integer power of a variable (often named x):

$$4x^5 \qquad -\frac{7}{2}t^{23} \qquad \pi z = \pi z^1 \qquad 111 = 111u^0$$

are all monomials, but

$$5\sqrt[3]{x} = 5x^{1/3} \qquad \frac{1}{t} = t^{-1}$$

are not.
A polynomial is a sum of finitely many monomials in the same variable:

$$-\frac{7}{2}x^{23} + 4x^5 + \pi \qquad 4t^3 - 5t^2 + 2t$$

are polynomials. To add polynomials, just combine like terms. To multiply two polynomials, make sure that every term of the first polynomial multiplies every terms of the second.

0.4.r1.

$$\begin{aligned}(x+3)(x^2-5x+4) =& x(x^2-5x+4) + 3(x^2-5x+4) \\ =& x \cdot x^2 + x \cdot (-5x) + x \cdot 4 \\ & \quad + 3 \cdot x^2 + 3 \cdot (-5x) + 3 \cdot 4 \\ =& x^3 - 5x^2 + 4x \\ & \quad + 3x^2 - 15x + 12 \\ =& x^3 - 2x^2 - 11x + 12\end{aligned}$$

Special case: when each of the two polynomials contains only two terms, you accomplish this by remembering the mnemonic FOIL.

0.4.r2. Expand (that is, multiply and collect like terms)

a. $(2x+3)(x-4)$ b. $(x-5)(2x+5)$ c. $(2x-5)(2x+5)$

d. $(4x+5)(6x-7)$ e. $(2x+3)(x-2)(3x-1)$ f. $(2x+3)(x^3-4x+1)$

To factor a polynomial is to rewrite it as a product of two or more polynomials of lower degree. When expanding or factoring, it pays to remember these **special products**:

(difference of squares)	$(a-b)(a+b) = a^2 - b^2$
(difference of cubes)	$(a-b)(a^2+ab+b^2) = a^3 - b^3$
(sum of cubes)	$(a+b)(a^2-ab+b^2) = a^3 + b^3$

(although you'll rarely see a sum or difference of cubes in this course) and these **perfect squares**:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

0.4.r3. Verify each of the special product and perfect square formula by expanding the left side.

Factoring a polynomial

1. Factor out any common factors.
2. Then try to factor the polynomial with one of these methods.
   a. If the polynomial has two terms, try using one of the **special products**.
   b. If the polynomial has three terms, try factoring it with **backwards-FOIL**. Heads-up: you may be looking at a perfect square. Can you recognize one?
   c. If the polynomial has four terms, try factoring it by **grouping**.
3. Are you done? Test whether you can factor the factors.

0.4.r4. $60x^3 - 5x^2 - 100x$ This polynomial has a common factor of $5x$:

$$60x^3 - 5x^2 - 100x = 5x(12x^2 - x - 20).$$

To factor the three-term polynomial, look for a factorization $(ax \pm b)(cx \pm d)$ that, when multiplied using FOIL, gives First $= 12x^2$, Last $= -20$, and Outside+Inside $= -x$:

$$5x(12x^2 - x - 20) = 5x(4x+5)(3x-4)$$

0.4.r5. Factor $8x^3 - 12x^2 + 10x - 15$. Since the polynomial has four terms, we factor the first two terms and the last two and hope to find a common factor.

$$(8x^3 - 12x^2) + (10x - 15) = 4x^2(2x-3) + 5(2x-3) = (4x^2+5)(2x-3).$$

The quadratic factor $4x^2 + 5$ is **irreducible**, meaning that it has no real zeros and cannot be factored further without complex numbers. The quadratic $x^2 - x + 1$ is also irreducible. We know this because its discriminant $b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0$.

Here's an interesting fact. Every polynomial with real coefficients can be factored as a product of linear and irreducible quadratic factors with real coefficients (although there is no exact method that produces the exact factorization of *all* polynomials). If we allow ourselves to use complex coefficients, then every polynomials can be factored as a product of linear factors only. Although complex numbers naturally arise when we factor quadratics, in this class we'll only be interested in factoring polynomials using real coefficients.

4. Factor the polynomial over the real numbers.

a. $x^2 - 16$ b. $x^2 - 9$ c. $50x - 2x^3$ d. $36u^2 - u^4$

e. $8 - 2x^2$ f. $16x^3 - 36x$ g. $4x^2 - 5$ h. $x^2 + 2x + 3$

5. Factor the polynomial over the real numbers.

a. $2x^2 - x - 1$ b. $2x^2 - 11x + 12$ c. $6x^2 - 13x + 5$

d. $4x^2 + 4x + 1$ e. $9 - 6y + y^2$ f. $8t^3 - 8t^2 + 2t$

g. $15x^2 + 17x - 4$ h. $15x^2 - 11x - 4$ i. $-6x^2 + 8x + 8$

6. Factor the polynomial over the real numbers.

a. $2x^3 - 3x^2 + 8x - 12$ b. $3x^3 + 6x^2 + 5x + 10$

c. $5x^3 - 3x^2 - 20x + 12$ d. $8x^3 + 4x^2 - 18x - 9$

e. $4x^3 - x^2 - 36x + 9$ f. $2x^3 - 3x^2 - 6x + 9$

g. $x^4 + 3x^3 - 8x - 24$ h. $27 - 81y + y^3 - 3y^4$

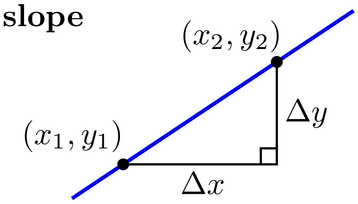
Answers

0.4.r2a. $2x^2 - 5x - 12$ 0.4.r2b. $2x^2 - 5x - 25$ 0.4.r2c. $4x^2 - 25$ 0.4.r2d. $20x^2 + 2x - 35$
0.4.r2e. $6x^3 - 5x^2 - 17x + 6$ 0.4.r2f. $2x^4 + 3x^3 - 8x^2 - 10x + 3$ 0.4.r5a. $(x+4)(x-4)$ 0.4.r5b. $(x-3)(x+3)$
0.4.r5c. $2x(5-x)(5+x)$ 0.4.r5d. $u^2(6-u)(6+u)$ 0.4.r5e. $2(2-x)(2+x)$
0.4.r5f. $4x(2x-3)(2x+3)$ 0.4.r5g. $(2x-\sqrt{5})(2x+\sqrt{5})$ 0.4.r5h. irreducible: does not factor further
0.4.r5a. $(2x+1)(x-1)$ 0.4.r5b. $(2x-3)(x-4)$ 0.4.r5c. $(3x-5)(2x-1)$ 0.4.r5d. $(2x+1)^2$
0.4.r5e. $(3-y)^2$ 0.4.r5f. $2t(2t-1)^2$ 0.4.r5g. $(3x+4)(5x-1)$ 0.4.r5h. $(15x+4)(x-1)$
0.4.r5i. $-2(3x+2)(x-2)$ 0.4.r5a. $(x^2+4)(2x-3)$ 0.4.r5b. $(3x^2+5)(x+2)$ 0.4.r5c. $(x-2)(x+2)(5x-3)$
0.4.r5d. $(2x+3)(2x-3)(2x+1)$ 0.4.r5e. $(x-3)(x+3)(4x-1)$ 0.4.r5f. $(x+\sqrt{3})(x-\sqrt{3})(2x-3)$
0.4.r5g. $(x+3)(x-2)(x^2+2x+4)$ 0.4.r5h. $(1-3y)(3+y)(9-9y+y^2)$

0.5: Lines and their equations

If a line contains the two points (x_1, y_1) and (x_2, y_2), then the **slope** of the line is the number

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}.$$



As the name suggests, the slope of the line will be the same for *any* two points on the line.

0.5.r1. The slope of the line through the points $(2, 8)$, $(5, 12)$ is $(12 - 8)/(5 - 2) = 4/3$. You'll get the same result whether you subtract $(2, 8)$ from $(5, 12)$ or subtract $(5, 12)$ from $(2, 8)$:

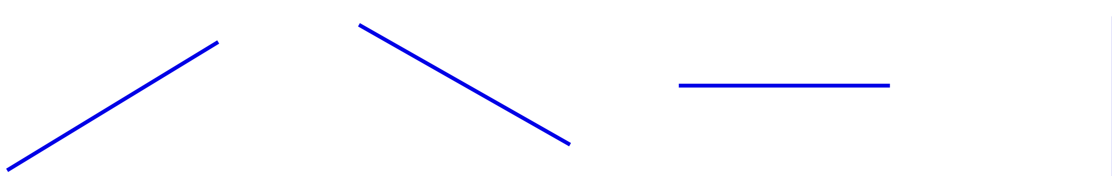
$$\frac{12-8}{5-2} = \frac{4}{3} \qquad\qquad \frac{8-12}{2-5} = \frac{-4}{-3} = \frac{4}{3}.$$

0.5.r2. Find the slope between the given points.

a. $(4, 2)$, $(-1, 2)$ b. $(1, 4)$, $(6, -2)$ c. $(3, -1)$, $(3, 4)$ d. $(2, -1)$, $(3, 4)$

The meaning of a line's slope

Whenever $\Delta x = 1$, slope equals Δy. That is, the slope of the line is the change in y along the line corresponding to a one-unit increase in x.
A line with positive slope rises from left to right; a line with negative slope falls from left to right. A horizontal line has zero slope; the slope of vertical line is undefined. We usually say that a vertical line has "no slope."



slope > 0 slope < 0 slope $= 0$ no slope

Two line are parallel if and only if they have the same slope (or both have no slope).
As long as neither is horizontal or vertical, two lines are perpendicular if and only their slopes are negative reciprocals, e.g., 2 and $-\frac{1}{2}$,

Equations of lines

Every line has many equaitons. For instance the three equations

$$y = 2x + 3 \qquad y - 3 = 2x \qquad y = 10x + 15$$

all describe the same line. Here are some useful ways of writing an equation of a line. The line passing through the point (x_0, y_0) with slope m has the **point-slope** equation

$$y - y_0 = m(x - x_0) \tag{0.5.r3}$$

The line with slope m and y-intercept $(0, b)$ has the **slope-intercept** equation

$$y = mx + b. \tag{0.5.r4}$$

Every horizontal line has the equation

$$y = c$$

for some constant c. Every vertical line has the equation

$$x = b$$

for some constant b.

0.5.r5. Find an equation of the given line. There is more than one answer.

a. The line passing through $(2, -1)$ and $(3, 4)$.

b. The line passing through $(-1, 6)$ with slope -2.

c. The line passing through $(5, 3)$ and $(5, 4)$.

d. The line passing through $(6, 3)$ parallel to the line in a.

e. The line passing through $(6, 3)$ perpendicular to the line in a.

f. The x-axis. (Hint: it's horizontal.)

Answers

0.5.r2a. 0 0.5.r2b. $-\frac{6}{5}$ 0.5.r2c. slope is undefined (no slope). 0.5.r2d. 5 0.5.r5a. $y + 1 = 5(x - 2)$ 0.5.r5b. $y - 6 = -2(x + 1)$ 0.5.r5c. $x = 5$. 0.5.r5d. $y - 3 = 5(x - 6)$ 0.5.r5e. $y - 3 = -\frac{1}{5}(x - 6)$ 0.5.r5f. $y = 0$

0.6: Linear equations in one variable

An equation in the variable x is **linear** if it could be rewritten $ax+b=c$ for some constants a, b, and c.

Fact 0.6.r1. *For any linear equation, one of these is true:*

- *The equation has no solution.*
- *The equation has exactly one solution.*
- *Every real number is a solution of the equation.*

To solve a linear equation, follow these steps. Always collect like terms to keep your work neat.

1. Distribute to get rid of any parentheses.
2. Add/subtract to get the variable terms on one side of the equation and the constant terms on the other.
3. Divide by the x-cooeficient to solve.

0.6.r2. Solve the equation $-10 = 5x - x - 18$.

First combine the x-terms to obtain

$$-10 = 4x - 18.$$

Add 18 to both sides to get

$$8 = 4x.$$

Finally, divide by 4 to reach the solution

$$2 = x.$$

0.6.r3. Solve the equation $6x - 2 = 4(x+1) - 1$.

Distribute the 4 and collect terms on the right side.

$$6x - 2 = 4x + 4 - 1 = 4x + 3.$$

Add 2 and subtract $4x$ from both sides to obtain

$$2x = 5.$$

Finally, divide by 2:

$$x = \frac{5}{2}.$$

0.6.r4. Solve the equation $\frac{6}{7}x - 2 = \frac{3}{5}(x+1) - 1$.

Carefully multipliy both sides by 7 to get rid of the fraction on the left:

$$6x - 14 = \frac{3}{5} \cdot 7(x+1) - 7$$

To get rid of the fraction on the right side, multiply both sides by 5:

$$5(6x - 14) = 3 \cdot 7(x+1) - 35,$$

which we can now solve as 0.6.r3.

0.6.r5. Solve the equation $2(x - 10) - 10 = 3x - (x + 4)$.

Distribute the 2 on the left side and the negative on the right side. Then collect like terms on each side:

$$2x - 10 - 10 = 3x - x - 4$$
$$2x - 20 = 2x - 4$$

Now, when we subtract $2x$ from each side,

$$-20 = -4.$$

There are no numbers x that make this equation true, so the original equation has no solutions.

0.6.r6. Solve the equation $2(x - 10) - 10 = 3x - (x + 20)$.

This time, the the steps we took in 0.6.r5 result in

$$2x - 20 = 2x - 20,$$

which is true regardless of the value of x. Every real number is a solution to the original equation.

An equation which is true for all real numbers is sometimes called an **identity**.

0.6.r7. Solve the equation.

a. $3x - 13 = 9x + 11$

b. $7x - 36 = 3x$

c. $20 + 7x = 12 + x$

d. $\dfrac{2}{3} - \dfrac{9}{2}x = \dfrac{5}{3}$

e. $\dfrac{2}{5} - \dfrac{1}{3}x = \dfrac{7}{15}$

f. $\dfrac{1}{2}x + 5 = \dfrac{1}{4}x + 6$

g. $2(x - 3) + 5 = (2x + 1) - (x + 4)$

h. $4(x - 2) + 5 = 3(x - 1) + 4$

i. $3 - (2 - 3x) = 5(x + 3)$

j. $\dfrac{1}{2}(4x - 1) = \dfrac{2}{3}(3x + 1) - 2$

k. $-(3 - x) + \dfrac{5}{3} = \dfrac{1}{9}(x - 2)$

l. $5 - (2x - 1) = \dfrac{1}{3}(x - 2) - x$

Answers

0.6.r7a. $x = -4$ 0.6.r7b. $x = 9$ 0.6.r7c. $x = -\frac{4}{3}$. 0.6.r7d. $x = -\frac{2}{9}$ 0.6.r7e. $x = -\frac{1}{5}$
0.6.r7f. $x = 4$ 0.6.r7g. $x = -2$ 0.6.r7h. $x = 4$ 0.6.r7i. $x = -7$. 0.6.r7j. no solutions.
0.6.r7k. $x = \frac{5}{4}$ 0.6.r7l. $x = 5$

0.7: Quadratic equations in one variable

An equation in the variable x is **quadratic** if it could be rewritten $ax^2 + bx + c = 0$ for some constants $a \neq 0$, b, and c.
The simplest way to solve a quadratic equation is to get a zero on one side and factor the other.

0.7.r1. Solve the equation $2x^2 - x = 3$.

Subtract 3 from both sides: $2x^2 - x - 3 = 0$. Factor the left side:

$$(2x-3)(x+1) = 0$$

Now use this fact:

The product of two numbers is zero if and only if one of the numbers is zero.

So, the solutions to $(2x-3)(x+1) = 0$ are the x-values that cause either $2x - 3$ or $x + 1$ to be zero:

$$\begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$

That is, the solutions to $2x^2 - x - 3 = 0$ are $x = \frac{3}{2}$ and $x = -1$.
For those times when we can't factor, there's the quadratic formula:

Quadratic formula 0.7.r2. *The solutions to $ax^2 + bx + c = 0$ are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

0.7.r3. Solve the equation $2x^2 - 5 = x$.

Rewrite the equation as $2x^2 - x - 5 = 0$. Since I don't see how to factor the left side, I'll use the quadratic formula with $a = 2$, $b = -1$, and $c = -5$. Solutions are

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2} = \frac{1 \pm \sqrt{41}}{4}$$

(This means that $2x^2 - x - 5 = 2(x - \frac{1+\sqrt{41}}{4})(x - \frac{1-\sqrt{41}}{4})$. No wonder I couldn't factor it!)

0.7.r4. Solve the equation $2x^2 - x + 1 = 0$.

I tried but cannot factor the left side, so I'll use the quadratic formula with $a = 2$, $b = -1$, and $c = 1$. This time, the solutions are

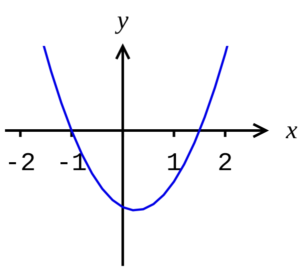
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (1)}}{2 \cdot 2} = \frac{1 \pm \sqrt{-7}}{4}.$$

There is no real number whose square is -7, so $\sqrt{-7}$ is not a real number. The equation $2x^2 - x + 1 = 0$ has no real solutions.

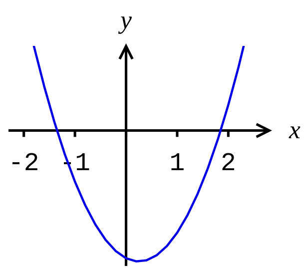
Solutions are intercepts

The solutions to $ax^2 + bx + c = 0$ are the x-intercepts of the graphs of $y = ax^2 + bx + c$.

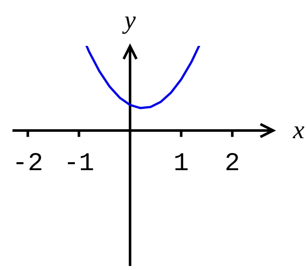
0.7.r1, 0.7.r3, and 0.7.r4, continued. See the graphs below.



$y = 2x^2 - x - 3$



$y = 2x^2 - x - 5$



$y = 2x^2 - x + 1$

The x-intercepts of $y = 2x^2 - x - 3$ are $x = -1$ and $x = 3/2 = 1.5$.

The x-intercepts of $y = 2x^2 - x - 5$ are $x = \frac{1-\sqrt{41}}{4} \approx -1.35$ and $x = \frac{1+\sqrt{41}}{4} \approx 1.85$.

The graph of $y = 2x^2 - x + 1$ has no x-intercepts.

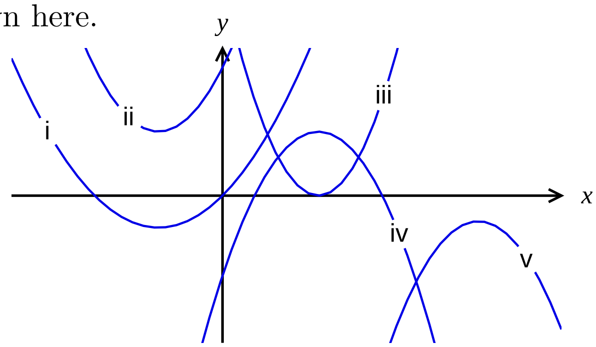
The discriminant and the number of solutions

Every quadratic equation has either 0, 1, or 2 real solutions, depending on the the value of the **discriminant** $b^2 - 4ac$ (the radicand in the quadratic formula).

If $b^2 - 4ac$ is	then $ax^2 + bx + c = 0$ has
positive	2 real solutions
zero	1 real solutions
negative	0 real solutions

0.7.r5. Find the value of c for which $y = 2x^2 - x + c = 0$ has exactly one solution.

0.7.r6. Determine whether $b^2 - 4ac$ is positive, negative, or zero for each of the five curves $y = ax^2 + bx + c$ shown here.



0.7.r7. Solve the equation.

a. $x^2 + 4x = 3$

b. $9x^2 + 6x + 3 = 0$

c. $2x^2 + 15 = -13x$

d. $\frac{4}{5}x^2 - \frac{13}{5}x + 2 = 0$

e. $x^2 - 5x - 24 = 0$

f. $4x^2 + 4x = -3$

g. $6x^2 - 5x = 6$

h. $3x^2 - 22x + 24 = 0$

i. $11 - 8x + 2x^2 = x^2$

j. $4x^2 - 12x + 9 = 0$

Answers

0.7.r5. $b^2 - 4ac = 1 - 8c$. Set $1 - 8c = 0$ to obtain $c = 1/8$. 0.7.r6. i positive. ii negative. iii zero. iv positive. v negative. 0.7.r7a. $x = -2 \pm \sqrt{7}$. 0.7.r7b. no real solutions. 0.7.r7c. $x = -5$ or $-3/2$. 0.7.r7d. $x =$ 2 or $5/4$. 0.7.r7e. $x = -3$ or 8. 0.7.r7f. no real solutions. 0.7.r7g. $x = -3/2$ or $2/3$. 0.7.r7h. $x = 4/3$ or 6. 0.7.r7i. $x = 4 \pm \sqrt{5}$. 0.7.r7j. $x = 3/2$.

0.8: Rational and radical equations in one variable

A **rational function** is a fraction whose numerator and denominator are both polynomials. The function is undefined at any x-value that causes the denominator to equal zero.

0.8.r1. $r(x) = \dfrac{x+3}{x^2-4}$ is a rational function. Factor the denominator to see where $r(x)$ is undefined.

$$r(x) = \frac{x+3}{x^2-4} = \frac{x+3}{(x+2)(x-2)}$$

$r(x)$ is undefined when $x = \pm 2$, since these x's cause the denominator $x^2 - 4 = 0$.

Rational equations

A **rational equation** is one involving rational functions. We can clear all the fractions in the equation by multiplying both sides by their least common denominator.

0.8.r2. To solve the equation $\dfrac{12}{x+1} = \dfrac{8}{x-2}$, first multiply both sides by $(x+1)(x-2)$ and simply each side by canceling common factors:

$$\begin{aligned}(x+1)(x-2)\frac{12}{x+1} &= \frac{8}{x-2}(x+1)(x-2)\\ (x-2)12 &= 8(x+1)\end{aligned}$$

Distribute and solve:

$$\begin{aligned}12x - 24 &= 8x + 8\\ 4x &= 32\\ x &= \frac{32}{4} = 8\end{aligned}$$

An x-value that causes any part of a rational equation to be undefined cannot be a solution to the equation.

0.8.r2, continued. The left side of

$$\frac{12}{x+1} = \frac{8}{x-2}$$

is undefined at $x = -1$. The right side is undefined at $x = 2$. Since the x-value found above, 8, is neither of these, it's safe to say that $x = 8$ *is* a solution of the orginal equation. Indeed, at $x = 8$:

$$\frac{12}{8+1} = \frac{12}{9} = \frac{4}{3}, \quad \text{and} \quad \frac{8}{8-2} = \frac{8}{6} = \frac{4}{3}$$

0.8.r3. Determine the unallowable x-values for the given equation; then solve the equation.

$$\frac{x-1}{5(x+1)} = \frac{x+7}{3(x+1)} \tag{0.8.r4}$$

The right and left sides of the equation (0.8.r4) do not allow $x = -1$, since this causes both denominators to equal zero.
Multiply both sides by the least common denominator of the two fractions in the equation, $15(x+1)$:

$$\begin{aligned} 15(x+1)\frac{x-1}{5(x+1)} &= 15(x+1)\frac{x+7}{3(x+1)} \\ 3(x-1) &= 5(x+7) \\ 3x-3 &= 5x+35 \\ -38 &= 2x \\ -19 &= x \end{aligned}$$

$x = -19$ is a solution to (0.8.r4) since $-19 \neq -1$.
When we cross-multiply (0.8.r4), we're actually multiplying both sides by $15(x+1)^2$:

$$\begin{aligned} 15(x+1)^2\frac{x-1}{5(x+1)} &= 15(x+1)^2\frac{x+7}{3(x+1)} \\ 3(x+1)(x-1) &= 5(x+1)(x+7) \end{aligned}$$

The solutions to this quadratic are $x = -19$ and $x = -1$. Since the original equation doesn't allow $x = -1$, we conclude that $x = -19$ is its only solution to (0.8.r4).

0.8.r5. Determine the unallowable x-values for the given equation. Then solve the equation.

a. $\dfrac{x-1}{x+2} = \dfrac{1}{2}$

b. $\dfrac{x+2}{x+6} = 2$

c. $\dfrac{1}{x+2} + \dfrac{2}{x+4} = \dfrac{4x}{(x+2)(x+4)}$

d. $\dfrac{x}{3+x} = \dfrac{1}{x-1}$

e. $\dfrac{2}{2x-3} = \dfrac{5}{4x+5}$

f. $\dfrac{6}{4x} = \dfrac{3}{x-5}$

g. $\dfrac{3}{8x} - \dfrac{5}{4} = \dfrac{1}{4x}$

h. $\dfrac{1}{x(x+2)} = \dfrac{4}{5x+1}$

Radical equations

A **radical equation** is one in which the variable appears under a radical. To solve, isolate the radical on one side of the equation and raise both sides to the appropriate power to eliminate the radical. If we raise both sides to an *even* power, we must check our answers in the original equation.

0.8.r6. Solve:

$$2 + \sqrt{8-x} = x$$

Subtract 2 from both sides before squaring both sides:

$$\sqrt{8-x} = x - 2$$
$$(\sqrt{8-x})^2 = (x-2)^2$$

Carefully expand the right side using FOIL:

$$8 - x = x^2 - 4x + 4$$

Get zero on the left side by add $x - 8$ to both sides. Then factor and solve:

$$0 = x^2 - 3x - 4 = (x-4)(x+1) \quad \Longrightarrow \quad x = 4 \text{ or } x = -1$$

Check both of these answers in the original equation. Remember that the symbol $\sqrt{}$ means the *positive* square root.

$$\begin{array}{rcl} x = 4 & \qquad & x = -1 \\ \sqrt{8-4} = 4-2 & & \sqrt{8+1} = -1-2 \\ \sqrt{4} = 2 \ \textbf{(true)} & & \sqrt{9} = -3 \ \textbf{(false)} \end{array}$$

Why must we check? There are *two* equations which both square to $8 - x = x^2 - 4x + 4$:

$$\sqrt{8-x} = x - 2 \qquad \text{and} \qquad \sqrt{8-x} = -(x-2)$$

We check our solutions to $8 - x = x^2 - 4x + 4$ in the original equation to see if they're solutions to the equation we want ($\sqrt{8-x} = x - 2$) or to the one we don't want ($\sqrt{8-x} = -(x-2)$).

0.8.r7. Solve the equation.

a. $\sqrt{7x+1} = x+1$

b. $\sqrt[3]{3x^2 - 14x - 4} = 1$

c. $2 + \sqrt{4x+37} = x$

d. $2 + \sqrt{4x+37} = 0$

e. $2 + \sqrt[3]{(x-1)(x+5)} = 0$

f. $\sqrt{28 - 4x - x^2} - 2 = x$

Answers

0.8.r5a. $x \neq -2$; $x = -4$. 0.8.r5b. $x \neq -6$; $x = -10$. 0.8.r5c. $x \neq -2$, $x \neq -4$; $x = 8$. 0.8.r5d. $x \neq -3$, $x \neq 1$; $x = 3$, $x = -1$. 0.8.r5e. $x \neq 3/2$, $x \neq -5/4$; $x = -5/2$. 0.8.r5f. $x \neq 0$, $x \neq 5$; $x = -5$. 0.8.r5g. $x \neq 0$; $x = 1/10$. 0.8.r5h. $x \neq 0, -2, -1/5$; $x = -1$, $x = 1/4$. 0.8.r7a. $x = 0$; $x = 5$. 0.8.r7b. $x = 5$; $x = -1/3$. 0.8.r7c. $x = 11$. 0.8.r7d. no solutions. 0.8.r7e. $= -1$, $x = -3$.

0.8.r7f. $x = 2$.

1.1: Functions

A **function** is a rule that assigns to each allowable **input** a unique **output**, or **value**. The set of all allowable inputs is called the function's **domain**; the set of all corresponding outputs is called the function's **range**. Typically, the domain and range are sets of real numbers.

A function is sometimes expressed by a formula and given a single letter as a name.

1.1.r1. Find the given values of the function $w(x) = x^2 - 2x - 1$

a. $w(0)$ b. $w(-1)$ c. $w(2)$

d. $w(2h)$ e. $w(h+1)$ f. $w(h)+1$

1.1.r2. Let f be the function given by $f(x) = (x+1)^2$. Since it is possible to add one to any number and square the result, every real number is an allowable input for $f(x)$. That is, the domain of $f(x)$ is the set of all real numbers, $(-\infty, \infty)$.

Generaly, it can be difficult to determine the range of a function, but in this case, we know the range of $f(x)$ is the set of all nonnegative numbers $[0, \infty)$, because for any nonnegative number a, there's at least one solution to $(x+1)^2 = a$.

1.1.r2, continued. Assuming $a \geq 0$, solve for x in the equation $(x+1)^2 = a$.

Domain rules

The domain of a function can be determined by thinking about the restrictions the function's formula put on the input.

1. If a function includes $\frac{\text{anything}}{A}$ or A^p for some number $p < 0$, then it requires $A \neq 0$.
2. If a function includes $\sqrt{B}$ or $\sqrt[p]{B}$ for some even number p, then it requires $B \geq 0$.

1.1.r3. Suppose that $g(x) = \sqrt{4-x} + x\sqrt{x+1}$. For $g(x)$ to be a real number it is necessary that $4 - x \geq 0$ and $x + 1 \geq 0$. Solve each inequality for x:

$$\begin{aligned} 0 \leq 4 - x \quad &\Longrightarrow \quad x \leq 4 \\ 0 \leq x + 1 \quad &\Longrightarrow \quad -1 \leq x \end{aligned}$$

The domain is all x for which

$$-1 \leq x \qquad \text{and} \qquad x \leq 4$$

That's precisely what we mean when we write the double inequality $-1 \leq x \leq 4$. The domain of g is the interval $[-1, 4]$.

1.1.r4. Suppose that $h(x) = \sqrt{4-x} + \frac{x}{\sqrt{x+1}}$. As in 1.1.r3, $4 - x$ must be ≥ 0. Because it appears under a square root, $x + 1$ must be ≥ 0, and since $\sqrt{x+1}$ is a deominator, $x + 1 \neq 0$. For $h(x)$ to be a real number requires that

$$-1 < x \qquad \text{and} \qquad x \leq 4$$

or $-1 < x \leq 4$. The domain of h is the interval $(-1, 4]$.

1.1.r5. Find the domain of the given function. Express your answer in terms of inequalities and again using interval notation.

a. $f(x) = \dfrac{x-1}{(x-2)(2x+1)}$

b. $w(-1)$

c. $u(x) = \sqrt{3x+5}$

d. $v(x) = \dfrac{x+1}{\sqrt{2x-1}}$

e. $w(x) = \dfrac{2}{1+\sqrt{x}}$

f. $r((x) = \dfrac{x+1}{2x-3}$

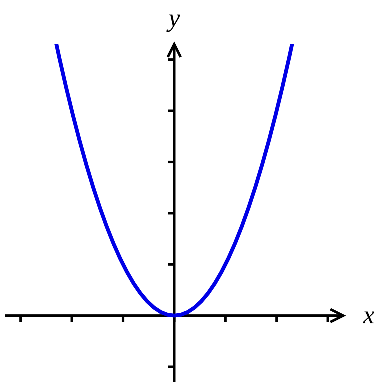
g. $s(x) = \begin{cases} 1+2x & \text{if } x < 1 \\ 1-\sqrt{x} & \text{if } 1 \leq x < 4. \end{cases}$

h. $t(x) = \begin{cases} x^2-1 & \text{if } -1 \leq x < 1 \\ \frac{1-2x}{x} & \text{if } 1 \leq x \leq 3. \end{cases}$

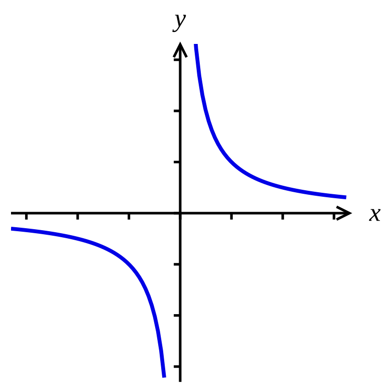
The graph of a function

The **graph** of a function $f(x)$ is the set of all points $(x, f(x))$ in the xy-plane where x is in the domain of $f(x)$. The graph is a geometric object that gives us a handy way to think about the function.

1.1.r6. The graphs of $f(x) = x^2$ and of $(x) = \dfrac{1}{x}$ appear below.



$y = x^2$



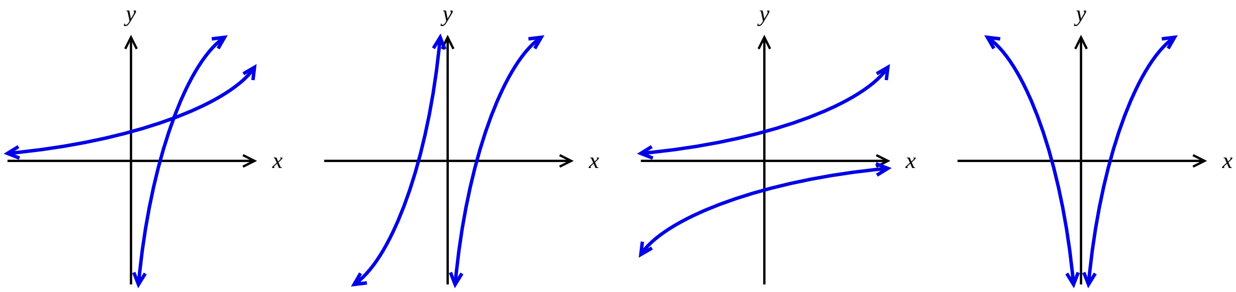
$y = \dfrac{1}{x}$

These graphs illustrate several things about these functions. For instance, x^2 is always ≥ 0 and equals 0 only if $x = 0$. The asymptotes of $y = \frac{1}{x}$ illustrate that the reciiprocal of a huge number is a tiny number (that is, close to zero), and the reciprocal of a tiny number is a huge number.

Note that no vertical line intersects the graph of either function more than once. This is because, by definition, a function f must have a unique value $f(x)$ for each x in its domain. If a vertical line were to intersect the graph of a function twice, that would mean that the function has two values $f(x)$ for some value of x.

Vertical line test 1.1.r7. *A curve in the xy-plane is the graph of a function and has an equation $y = f(x)$ if and only if no vertical line intersects the curve more than once.*

1.1.r8. Determine which of the given curves is the graph of a function.



a. b. c. d.

Linear functions

A **linear** function is one that can be written in the form $\ell(x) = ax + b$ for some constants a and b. A function is linear if and only if its graph is a line. When we solve for y in the point-slope form of the equation of a line (0.5.r3), the result is a linear function in the form $\ell(x) = m(x - x_0) + y_0$.

1.1.r9. Find the linear function described in each part. (Compare with 0.5.r5.)

a. $r(0) = 3$ and $r(4) = 6$.

b. $s(1) = 0$ and $s(4) = 2$.

c. $t(12) = 9$ and $t(13) = 7$.

d. $\tau(0) = 5$. Every 2-unit increase in x causes $\tau(x)$ to decrease by 6 units.

.

Answers

1.1.r1a. $0^2 - 2\cdot 0 - 1 = -1$. 1.1.r1b. 2. 1.1.r1c. -1. 1.1.r1d. $(2h)^2 - 2\cdot 2h - 1$, or $4h^2 - 4h - 1$. 1.1.r1e. $(h+1)^2 - 2(h+1) - 1$, or $h^2 - 2$. 1.1.r1f. $h^2 - 2h - 1 + 1$, or $h^2 - 2h$. 1.1.r2. $x = -1 + \sqrt{a}$. 1.1.r5a. $x \neq 2$ and $x \neq -1/2$; $(-\infty, -1/2) \cup (-1/2, 2) \cup (2, \infty)$. 1.1.r5b. 2. 1.1.r5c. $x \geq -5/3$; $[-5/3, \infty)$. 1.1.r5d. $x > 1/2$; $(1/2, \infty)$. 1.1.r5e. (Hint: $1 + \sqrt{x}$ is never zero.) $x \geq 0$; $[0, \infty)$. 1.1.r5f. $x \neq 3/2$; $(-\infty, 3/2) \cup (3/2, \infty)$. 1.1.r5g. $s(x)$ is defined for all $x < 4$; $(-\infty, 4)$. 1.1.r5h. $t(x)$ is defined for all $-1 \leq x \leq 3$; $[-1, 3]$. 1.1.r8. a. not a function. b. function. c. not a function. d. function. 1.1.r9a. $r(x) = \frac{3}{4}x + 3$. 1.1.r9b. $s(x) = \frac{2}{3}(x - 1)$. 1.1.r9c. $t(x) = -2(x - 12) + 9$. 1.1.r9d. $\tau(0) = 5 - 3x$

1.2: Operations with functions

This section is about how we combine functions using the arithmetic operations and composition to create other functions.

Addition, subtraction, multiplication, and division of functions

1.2.r1. If $f(x) = x^2 - 6x + 8$ and $g(x) = x^2 - 4$ find the following:

a. $f(x) + g(x)$ b. $f(x) - g(x)$ c. $f(x) \times g(x)$ d. $f(x) \div g(x)$

Solutions:
a. $f(x) + g(x) = (x^2 - 6x + 8) + (x^2 - 4)$. Eliminate the parentheses and collect like terms

$$x^2 - 6x + 8 + x^2 - 4 = 2x^2 - 6x + 4.$$

That answers the question, but in a larger problem it might be useful to factor a trinomial like this one. To do so, begin by factoring out the common factor and then use backwards FOIL:

$$2x^2 - 6x + 4 = 2(x^2 - 3x + 2) = 2(x - 1)(x - 2)$$

b. $f(x) - g(x) = (x^2 - 6x + 8) - (x^2 - 4)$. Be careful to distribute the minus with eliminating the parentheses:

$$(x^2 - 6x + 8) - (x^2 - 4) = x^2 - 6x + 8 - x^2 + 4 = -6x + 12$$

which factors as $-6(x - 2)$.
c. $f(x) \times g(x) = (x^2 - 6x + 8)(x^2 - 4)$. If you need to expand this, just make sure every terms in the first "( )" multiplies every terms in the second. It's easilest to use multiple lines and maintain like terms in columns:

$$\begin{aligned}(x^2 - 6x + 8)(x^2 - 4) &= (x^2 - 6x + 8)x^2 + (x^2 - 6x + 8)(-4) \\ &= x^4 - 6x^3 + 8x^2 \\ &\qquad\qquad - 4x^2 + 24x - 32 \\ &= x^4 - 6x^3 + 4x^2 + 24x - 32\end{aligned}$$

If, instead, you need to factor $f(x) \times g(x)$, try to factor both $f(x)$ and $g(x)$:

$$\begin{aligned}f(x)g(x) = (x^2 - 6x + 8)(x^2 - 4) &= (x - 2)(x - 4)(x - 2)(x + 2) \\ &= (x - 2)^2(x - 4)(x + 2)\end{aligned}$$

d. $f(x) \div g(x) = (x^2 - 6x + 8) \div (x^2 - 4) = \frac{x^2-6x+8}{x^2-4}$. We don't need parentheses around the numberator or denominator because the long fraction bar performs the same function. (See 0.2.r8.) It is often useful to factor and cancel any common factor.

$$\frac{x^2 - 6x + 8}{x^2 - 4} = \frac{(x - 2)(x - 4)}{(x - 2)(x + 2)} = \frac{x - 4}{x + 2}.$$

1.2.r2. Find (**i**) $f(x)+g(x)$, (**ii**) $f(x)-g(x)$, (**iii**) $f(x)\times g(x)$, and (**iv**) $f(x)\div g(x)$ for the given $f(x)$ and $g(x)$.

a. $f(x)=\sqrt{x}$; $g(x)=x+1$

b. $f(x)=2x+1$; $g(x)=x-3$

c. $f(x)=\sqrt{x+3}$; $g(x)=x+3$

d. $f(x)=3x^2+2x-5$; $g(x)=x^2-9x+8$

Composition of functions

To compose two functions is to use the output of one as the input of the other.

1.2.r3. Find $f\big(g(x)\big)$ and $g\big(f(x)\big)$ if $f(x)=x^2-3x+1$ and $g(x)=2x-5$

To find $f\big(g(x)\big)$, we have to replace the "x" in

$$f(x)=x^2-3x+1$$

with $g(x)$, like so:

$$\begin{aligned} f\big(g(x)\big) &= \big(g(x)\big)^2-3\big(g(x)\big)+1 \\ &= \big(2x-5\big)^2-3\big(2x-5\big)+1 \end{aligned}$$

To perform this step, some students find it helpful to begin by replacing "x" in

$$f(x)=x^2-3x+1$$

with an empty pair of parentheses

$$f(\quad)=(\quad)^2-3(\quad)+1$$

and then filling those with $g(x)$:

$$f\big(2x-5\big)=\big(2x-5\big)^2-3\big(2x-5\big)+1$$

In a larger problem, it might help to simplify $f\big(g(x)\big)$ by rewriting the above without parentheses, like so:

$$\begin{aligned} f\big(g(x)\big) &= \big(4x^2-20x+25\big)-6x+15+1 \\ &= 4x^2-26x+41 \end{aligned}$$

We find $g\big(f(x)\big)$ similarly:

$$\begin{aligned} g(x) &= 2x-5 \\ g(\quad) &= 2(\quad)-5 \\ g\big(f(x)\big)=g\big(x^2-3x+1\big) &= 2\big(x^2-3x+1\big)-5 \\ &= 2x^2-6x+2-5 \\ &= 2x^2-6x-3 \end{aligned}$$

1.2.r4. Find (**i**) $u\big(w(x)\big)$ and (**ii**) $w\big(u(x)\big)$ for the given functions $u(x)$ and $w(x)$.

a. $u(x) = \sqrt{x-1}$; $w(x) = 2x+1$

b. $u(x) = 2x+1$; $w(x) = 3x-4$

c. $u(x) = \sqrt{x+2}$; $w(x) = x^2+x$

d. $u(x) = 2x+1$; $w(x) = 1 - \dfrac{1}{2}x^2$

1.2.r5. Find two functions so that $f(x)$ and $g(x)$ so that

$$f\big(g(x)\big) = \sqrt[3]{x^2+2x+2}$$

There are many correct answers, but $g(x)$ has to be a first part of the calculation performed in $f\big(g(x)\big)$ and then $f(\ \ )$ must be the operations done to $g(x)$ to produce the final output. If we let

$$g(x) = x^2+2x+2 \qquad \text{and} \qquad f(\qquad) = \sqrt[3]{\qquad}$$

then

$$f(g(x)) = f\big(x^2+2x+2\big) = \sqrt[3]{x^2+2x+2}$$

as desired. Therefore,

$$g(x) = x^2+2x+2 \qquad \text{and} \qquad f(x) = \sqrt[3]{x}$$

is one working answer. (So is $g(x) = x^2+2x$ and $f(x) = \sqrt[3]{x+2}$.)

1.2.r6. Find two functions $r(x)$ and $s(x)$ so that the given function is $r\big(s(x)\big)$. There's more than correct answer to each part.

a. $(2-\sqrt{x})^3$

b. $\dfrac{10}{x^2-1}$

c. $\sqrt{1+x} - (1+x)^2 + 4$

d. $x^2 - 4x + 4$

Hint: d is a perfect square.

Answers

1.2.r2a. **i**. $\sqrt{x}+x+1$. **ii**. $\sqrt{x}-x-1$. **iii**. $\sqrt{x}(x+1)$, or $x^{3/2}+x^{1/2}$. **iv**. $\frac{\sqrt{x}}{x+1}$ or $\sqrt{x}(x+1)^{-1}$. 1.2.r2b. **i**. $3x-2$. **ii**. $x+4$. **iii**. $(2x+1)(x-3)$, or $2x^2-5x-3$. **iv**. $\frac{2x+1}{x-3}$. 1.2.r2c. **i**. $\sqrt{x+3}+x+3$. **ii**. $\sqrt{x+3}-x-3$. **iii**. $\sqrt{x+3}(x+3)$, or $(x+3)^{3/2}$. **iv**. $\frac{\sqrt{x+3}}{x+3}$ or $(x+3)^{-1/2}$ or $\frac{1}{\sqrt{x+3}}$. 1.2.r2d. **i**. $4x^2-7x+3$, or $(4x-3)(x-1)$. **ii**. $2x^2+11x-13$ or $(2x+13)(x-1)$. **iii**. $(3x^2+2x-5)(x^2-9x+8)$, or $(x-1)^2(3x+5)(x-8)$. **iv**. $\frac{3x^2+2x-5}{x^2-9x+8}$ or $\frac{3x+5}{x-8}$. 1.2.r4a. **i**. $\sqrt{2x}$. **ii**. $2\sqrt{x-1}+1$. 1.2.r4b. **i**. $6x-7$. **ii**. $6x+1$. 1.2.r4c. **i**. $\sqrt{x^2+x+2}$. **ii**. $x+2+\sqrt{x+2}$. 1.2.r4d. **i**. $3-x^2$. **ii**. $-4x^2-4x$, or $-4x(x-1)$. 1.2.r6a. $r(x) = x^3$; $s(x) = 2-\sqrt{x}$ 1.2.r6b. $r(x) = \frac{10}{x}$; $s(x) = x^2-1$ 1.2.r6c. $r(x) = \sqrt{x} - x^2 + 4$; $s(x) = 1+x$ 1.2.r6d. $r(x) = x^2$; $s(x) = x-2$

1.3: Functions from economics

A merchant has a product for sale. The number of units of their product that they are able to sell will depend on the price they set for the prouduct. Generally, the higher the price, the fewer units they can expect to sell. (When we count their sales, it's always in units of product per unit of time, like cars per month or pizzas per week.)

These functions appear often in our course.

x : The number of units sold.

$D(x)$: The (highest) price at which buyers in the market are willing to buy x units. $D(x)$ is called the **demand** function in our text.

$C(x)$: The merchant's **cost** of obtaining x units of their product.

$C(0)$: The merchant's **fixed cost**.

$\bar{C}(x) = \dfrac{C(x)}{x}$: The merchant's **average cost** per unit of obtaining x units of their product. (Average cost doesn't appear in the homework until section 2.9.)

$R(x) = xD(x)$: The merchant's **revenue** when they sell x units. Revenue equals price times quantity.

$P(x) = R(x) - C(x)$: The merchant's **profit** when they obtain and sell x units. Profit equals revenue minus cost.

Break-even point: An x-value at which profit $P(x) = 0$.

1.3.r1. A bagel bakery has fixed costs of \$300 per day. Each bagel costs \$0.75 to produce. What is the total cost to the bakery of producing x bagels in a day?

Solution: The bakery's total cost of producing x bagels in a day is \$300 plus \$0.75 times the number of bagels:

$$C(x) = 300 + 0.75x \quad \text{dollars}$$

1.3.r1, continued. Find the bakery's revenue and profit functions if the demand function for the bagels produced at the bakery is

$$D(x) = 6 - \frac{1}{1000}x$$

Find any break-even points x for the bakery.

Solution: Revenue is price time quantity:

$$R(x) = xD(x) = x(6 - \frac{1}{1000}x) = 6x - \frac{1}{1000}x^2$$

Profit is revenue minus cost:

$$P(x) = xD(x) - C(x) = 6x - \frac{1}{1000}x^2 - (300 + 0.75x) = -300 + 5.25x - \frac{1}{1000}x^2$$

To find the break-even points, set $P(x) = 0$ and solve. Although we could solve using the quadratic formula, it's easiest to find the solutions using a graphing calculator, for instance https://www.desmos.com/calculator/cmxzdmlaqs. With a grapher, we find the solutions

$$x = 57.78 \text{ and } x = 5192.22$$

1.3.r1, continued. Suppose the bakery made and sold 2000 bagels on Monday. On Tuesday it lowered its price and sold 4000 bagels. Find its price, revenue, and profit both days.

Solution: On Monday, when the bakery sold 2000 bagels, it was at the price of $D(2000) = 6 - \frac{1}{1000}2000 = 4$ dollars. On that day, revenue was

$$R(2000) = 2000 \cdot 4 = 8000$$

dollars and its profit was

$$P(2000) = -300 + 5.25 \cdot 2000 - \frac{1}{1000} \cdot 2000^2 = 6200$$

dollars.
To sell 4000 bagels Tuesday, the bakery had to lower its price to $D(4000) = 6 - \frac{1}{1000}4000 = 2$ dollars. Revenue that day was

$$R(4000) = 4000 \cdot 2 = 8000$$

dollars, and profit was

$$P(4000) = -300 + 5.25 \cdot 4000 - \frac{1}{1000} \cdot 4000^2 = 4700$$

dollars.
Note that selling more bagels at a lower price does not necessarily mean a greater profit that day.

end example 1.3.r1

In 1.3.r1, both the cost function and the demand function are **linear** (p. 25 of these notes).

1.3.r2. A manufacturer of jogging strollers can produce 250 strollers in a week at a cost of \$3,600 and 400 strollers in a week at a cost of \$5,400. Find the cost function $C(x)$ of producing x strollers in a week, assuming the function is linear. What is the manufacturer's fixed weekly cost?

1.3.r3. A pencil maker can sell 10 cases of pencils a day at \$45 per case. To sell an additional case per day required a \$2 decrease in price. Find the demand function for pencils, assuming it is linear. What is the manufacturer's revenue function?

Answers

1.3.r2. $C(x) = 3600 + 12(x - 250)$. Fixed cost is $C(0) = 3600 + 12(-250) = 600$ dollars. 1.3.r3. $D(x) = 45 - 2(x - 10)$. $R(x) = xD(x) = 45x - 2x(x - 10)$, or $-2x^2 + 65x$.

2.1: One-sided limits

2.1.r1. Evaluate the limit of $\ell(x) = \dfrac{x^2-1}{|x-1|}$ as x goes to 1 from the right and again from the left.

Solution: Using a spreadsheet, I calculated this table of values of $\ell(x)$ near $x = 1$.

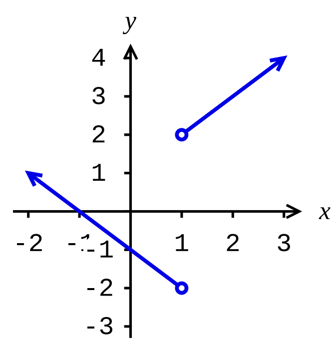
x	$\dfrac{x^2-1}{\lvert x-1\rvert}$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001
1.00001	2.00001

x	$\dfrac{x^2-1}{\lvert x-1\rvert}$
0.9	-1.9
0.99	-1.99
0.999	-1.999
0.9999	-1.9999
0.99999	-1.99999

Although $\ell(1)$ is not defined, it appears that as x approaches 1 from numbers greater than 1, the corresponding value of $\ell(x)$ approaches 2. As x approaches 1 from numbers less than than 1, it appears that $\ell(x)$ approaches -2.
If that guess is correct, then we'd write

$$\lim_{x\to 1^+} \ell(x) = 2 \quad \text{and} \quad \lim_{x\to 1^-} \ell(x) = -2$$

The first is read "the **limit** of $\ell(x)$ as x goes to 1 from the right equals 2," and the second is read "the **limit** of $\ell(x)$ as x goes to 1 from the left equals -2."



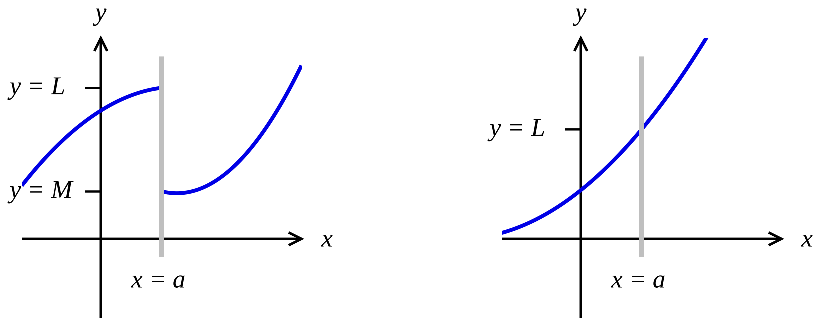
It turns out that the graph of ℓ consists of two straight line segments. To indicate that $\ell(1)$ is undefined, I added open circles at the the places where the line segments end at $x = 1$. (I added arrows to the other ends because, in fact, the graph continues in those directions offscreen.) You can produce a similar graph on Desmos:

https://www.desmos.com/calculator/cpxeexfggt

In 2.1.r1, $\lim_{x\to 1^+} \ell(x)$ is the altitude 2 of the point at the end of the segment of the graph which lies to the right of $x = 1$. The limit from the left, $\lim_{x\to 1^-} \ell(x)$ is the altitude of the point at the end of the segment of the graph which lies to the left of $x = 1$.

end example 2.1.r1

Imagine the curve $y = f(x)$ divided by the line $x = a$ into a segment to the right of a and another segment to the left of a. The left- and right-hand limits of $f(x)$ at a are the y-coordinates of the endpoints of those segments at $x = a$. Note that the limits don't depend on whether those endpoints are drawn with a • or with a ◦.
This is easier to picture when the limits from the left and the right are different, as in the figure on the left.



$$\lim_{x\to a^-} f(x) = L \qquad\qquad \lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a^+} f(x) = M \qquad\qquad \lim_{x\to a^+} f(x) = L$$

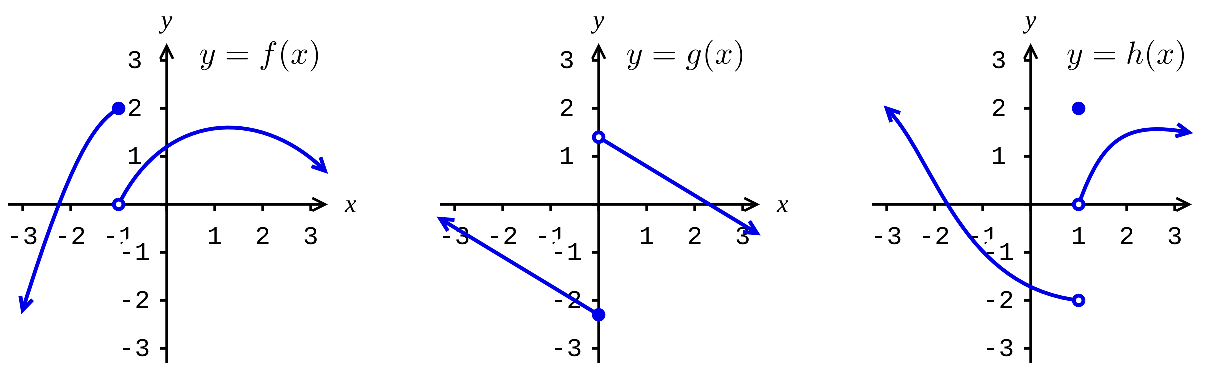
Figure 2.1.r2

But the limits from the left and from the right can be the same, as in the figure on the right.

2.1.r3. Estimate the following limits.

a. $\lim\limits_{x\to -1^+} f(x)$ b. $\lim\limits_{x\to 0^+} g(x)$ c. $\lim\limits_{x\to 1^+} h(x)$

d. $\lim\limits_{x\to -1^-} f(x)$ e. $\lim\limits_{x\to 0^-} g(x)$ f. $\lim\limits_{x\to 1^-} h(x)$

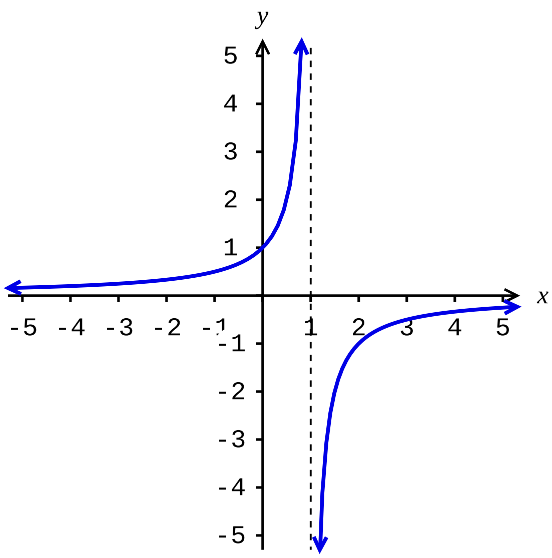


Infinite limits

"∞" is not a number. Rather, it is a symbol that we use in some situations to express growth beyond any finite limit.

2.1.r4. The graph of $y = \dfrac{1}{1-x}$ and tables of its values appear below. Notice that as x approaches 1 from the left, $\dfrac{1}{1-x}$ doesn't approach any finite number, but instead grows greater and greater without bound. As x approaches 1 from the right, $\dfrac{1}{1-x}$ grows less and less. To express this, we say

$$\lim_{x\to 1^-} \frac{1}{1-x} = \infty \quad \text{and} \quad \lim_{x\to 1^+} \frac{1}{1-x} = -\infty$$



x	$\dfrac{1}{1-x}$
1.1	-10
1.01	-100
1.001	$-1,000$
1.0001	$-10,000$
1.00001	$-100,000$

x	$\dfrac{1}{1-x}$
0.9	10
0.99	100
0.999	$1,000$
00.9999	$10,000$
0.99999	$100,000$

Tip: One-sided limits of the form "$\frac{\text{nonzero}}{0}$" are infinite (except for some unusual examples not seen in this class).

2.1.r5. Use graphs to determine which of these one-sided limits are ∞ and which are $-\infty$.

a. $\displaystyle\lim_{x\to 2^+} \frac{x-3}{\sqrt{x-2}}$ b. $\displaystyle\lim_{x\to 5^-} \frac{2}{x-5}$ c. $\displaystyle\lim_{x\to 3^+} \frac{x^2}{x-3}$

2.1.r6. What limit is suggested by each table? In each, your answer should look like "$\displaystyle\lim_{x\to a^\pm} f(x) = b$." for some a, b, and f.

x	$u(x)$
2.4	2.1
2.49	2.01
2.499	2.001
2.4999	2.0001

x	$v(x)$
2.1	55
2.01	549
2.001	5490
2.0001	54900

x	$w(x)$
4.9	-5.9
4.99	-5.95
4.999	-5.994
4.9999	-5.9993

Answers

2.1.r3a. 0. 2.1.r3b. 1.4, approximately. 2.1.r3c. 0. 2.1.r3d. 2. 2.1.r3e. -2.2, approximately. 2.1.r3f. -2. 2.1.r5. a $-\infty$ b $-\infty$. c ∞. 2.1.r6. $\lim_{x\to 2.5^-} u(x) = 2$. $\lim_{x\to 2^+} v(x) = \infty$. $\lim_{x\to 5^-} w(x) = -6$.

2.2: Limits

If the one-sided limits

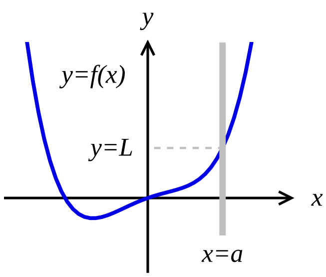
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

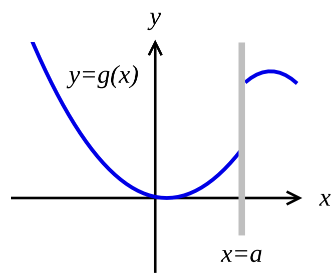
(as in the right half of Figure 2.1.r2), then we say the **limit** of $f(x)$ as x goes to a equals L, written

$$\lim_{x\to a} f(x) = L.$$

In other words, we say the limit of $f(x)$ exists as $x \to a$ if both one-sided limits at a exists and are equal.

2.2.r1. In the figure below, the limits of $f(x)$ as $x \to a^+$ and as $x \to a^-$ exist and are equal. Therefore $\lim_{x\to a} f(x)$ also exists. The limit equals the y-value L in the figure.

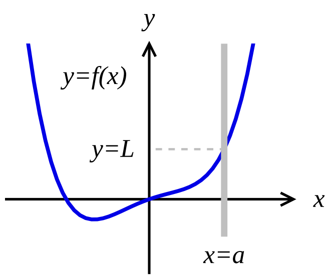


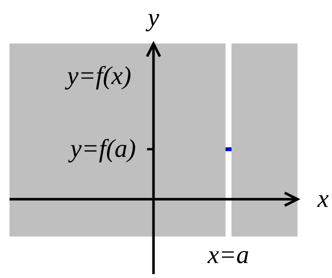


On the other hand the limits of $g(x)$ as $x \to a^+$ and as $x \to a^-$ exist but are unequal. Therefore $\lim_{x\to a} g(x)$ does not exist.

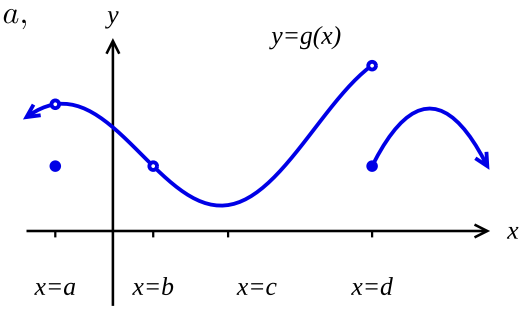
Limit values versus function values

The $\lim_{x\to a} f(x)$ is based on the values of f at x's near a, ignoring what the function does at a itself. In contract, the function value $f(a)$ is based solely on what f does at a, and ignores all other values:

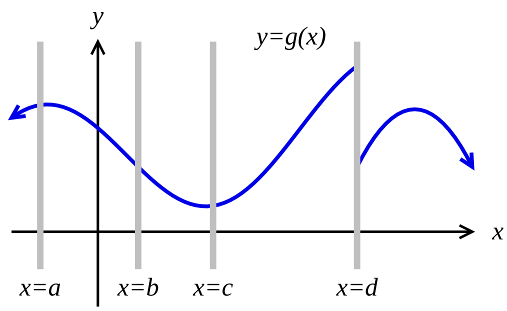


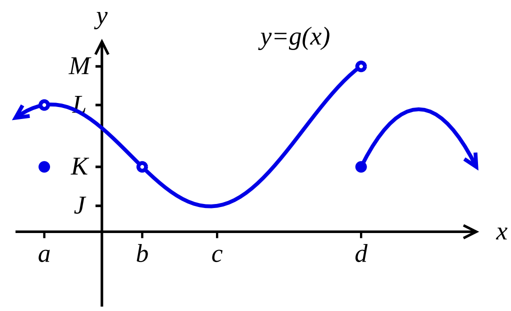


2.2.r2. Compare function values and limits values at a, b, c, and d for the function $g(x)$ graphed at right.



Solution:
When we look at y-values on the graph of $g(x)$ at x near but not equal a, b, c, and d as in below left, we see that the limit exists at a, b, and c, but not at d. In contrast, when we focus on altitudes of the curve at the four points, we see that $g(x)$ exists at $x = a$, c, and d, but not at b.





After drawing and labeling hash marks at the relevant y-values as in above right,

$$g(a) = K, \text{ but } \lim_{x \to a} g(x) = L.$$

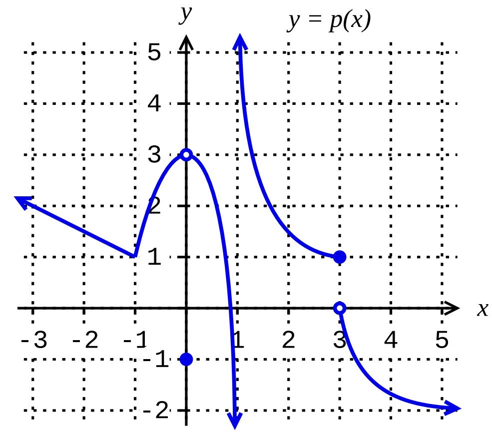
$$g(b) \text{ does not exist, but } \lim_{x \to b} g(x) = K.$$

$$g(c) \text{ and } \lim_{x \to c} g(x) \text{ exist and are equal to } J.$$

$$g(d) = K, \text{ but } \lim_{x \to d} g(x) \text{ does not exist.}$$

end example 2.2.r2

2.2.r3. Use the graph of $p(x)$ to find the following. Your response to each part should be a number, infinity, -infinity, or DNE.



a. $\lim_{x \to 0} p(x)$

b. $\lim_{x \to 3} p(x)$

c. $\lim_{x \to 3^+} p(x)$

d. $p(0)$

e. $p(3)$

f. $\lim_{x \to -1} p(x)$

g. $\lim_{x \to 1^-} p(x)$

h. $\lim_{x \to 1} p(x)$

i. $p(-1)$

j. $\lim_{x \to 2} p(x)$

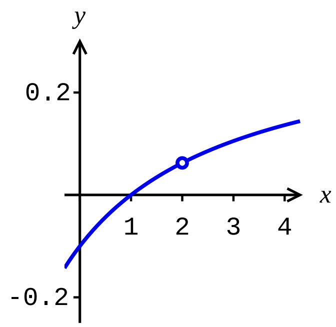
Finding limits algebraically

2.2.r4. Evaluate the limit $\lim_{x\to 2} \dfrac{x^2-3x+2}{3x^2+4x-20}$.

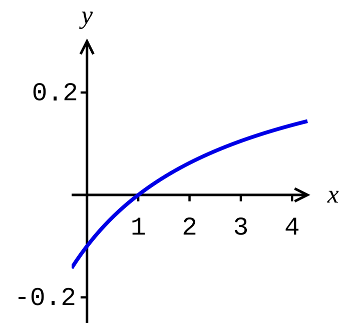
Solution: Observe that the numerator and denominator both equal zero at $x=2$. That means $(x-2)$ must be a factor of each. Factor and cancel the common factor:

$$\frac{x^2-3x+2}{3x^2+4x-20} = \frac{(x-2)(x-1)}{(x-2)(3x+10)} =^* \frac{x-1}{3x+10}$$

*except at $x=2$, where the left side does not exist.



The graph of $y = \frac{(x-2)(x-1)}{(x-2)(3x+10)}$.



The graph of $y = \frac{x-1}{3x+10}$.

The limit is the height of the hole in the graph on the left, and that equals be the height of the graph on the right at $x=2$:

$$\lim_{x\to 2} \frac{x^2-3x+2}{3x^2+4x-20} = \lim_{x\to 2} \frac{x-1}{3x+10} = \frac{2-1}{3\cdot 2+10} = \frac{1}{16}.$$

end example 2.2.r4

2.2.r5. Evaluate the limit algebraically.

a. $\lim_{x\to -1} \dfrac{x+1}{4x^2+3x-1}$

b. $\lim_{x\to -3} \dfrac{2x^2-18}{x^2+x-6}$

c. $\lim_{x\to 2} \dfrac{x^2-5x+6}{2x^2+x-10}$

d. $\lim_{x\to 5} \dfrac{x^2-25}{x^2-7x+10}$

Answers

2.2.r3a. 3 2.2.r3b. DNE 2.2.r3c. 0 2.2.r3d. -1 2.2.r3e. 1 2.2.r3f. 1 2.2.r3g. $-\infty$
2.2.r3h. DNE 2.2.r3i. 1 2.2.r3j. 1.5 2.2.r5a. $-\frac{1}{3}$ 2.2.r5b. $\frac{12}{5}$ 2.2.r5c. $-\frac{1}{9}$ 2.2.r5d. $\frac{10}{3}$

2.3: Properties of limits

These rules allow us to calculate limits more easily.

I. Examples of some elementary continuous functions.

1. $\lim_{x\to a} x = a$
2. $\lim_{x\to a} c = c$

II. Combination laws. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are finite, then:

3. $\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\big(f(x)-g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\big(f(x)\times g(x)\big) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
6. $\lim_{x\to a}\big(f(x)\div g(x)\big) = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$ (except in the case of division by zero).

If c is a constant, then

7. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

If r is a rational number, then, except in case of division by zero or roots that do not exist,

8. $\lim_{x\to a}\big(f(x)\big)^r = \big(\lim_{x\to a} f(x)\big)^r$

As a consequence of rules 1. and 9.,

9. $\lim_{x\to a} x^r = a^r$, (except in the case of division by zero or roots that do not exist)

10. If $p(x)$ is any polynomial and a is any number, then $\lim_{x\to a} p(x) = p(a)$.

11. If $r(x)$ is any rational function (i.e., the ratio of two polynomials) and a is any number, then $\lim_{x\to a} r(x) = r(a)$, provided $r(a)$ exists.

2.3.r1. The rational function $\dfrac{x-2}{(x-1)^2}$ is defined for all $x \neq 1$. Its value at $x = 3$ is $\dfrac{3-2}{(3-1)^2} = \dfrac{1}{4}$, so by 11,

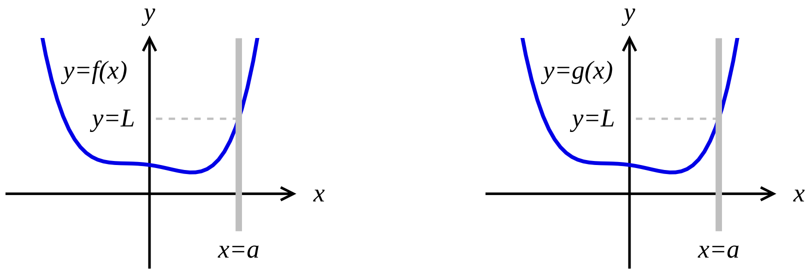
$$\lim_{x\to 3}\frac{x-2}{(x-1)^2} = \frac{1}{4}.$$

end example 2.3.r1

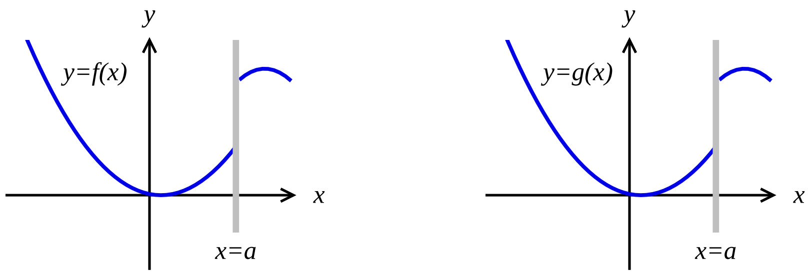
The most commonly used limit law of all:

12. If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are the same, whether both exist (as in 2.3.r2) or both fail to exist (2.3.r3).

2.3.r2.



2.3.r3.



2.3.r4. Evaluate the limit, if it exists: $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-x-2}$

Solution:

Factor and cancel the common factor.

$$\frac{x-2}{x^2-x-2} = \frac{x-2}{(x-1)(x-2)} =^* \frac{1}{x-1}$$

(* except at $x = 2$, where $\frac{1}{x-1}$ exists but $\frac{x-2}{(x-1)(x-2)}$ does not.) By 12,

$$\lim_{x\to 2} \frac{x-2}{x^2-x-2} = \lim_{x\to 2} \frac{1}{x-1}.$$

Since the rational function $\dfrac{1}{x-1}$ exists at $x = 2$, its limit equals its function value there:

$$\lim_{x\to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1.$$

end example 2.3.r4

2.3.r5. Evaluate the limits:

a. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{3x^2-6x}$

b. $\lim\limits_{x\to -2} \dfrac{2x^2+5x+2}{x^2-8x-14}$

Tip: In the expression

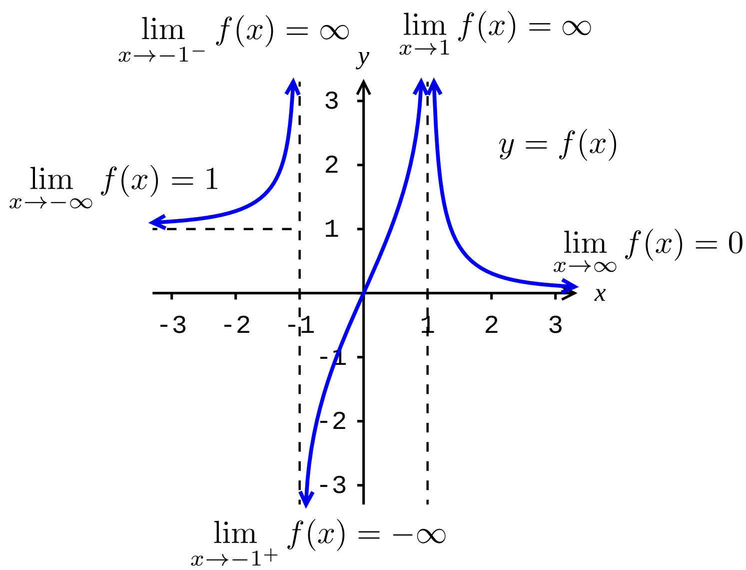
$$\lim_{x \to \square} f(x) = \triangle$$

$\square$ is an x-value, and $\triangle$ is a y-value.

Limits at infinity, infinite limits, and horizontal and vertical asymptotes.

If $\lim\limits_{x \to a^{\pm}} f(x) = \pm\infty$, then the vertical line $x = a$ is an asymptote to the graph of $f(x)$.

If $\lim\limits_{x \to \pm\infty} f(x) = c$, then the horizontal line $y = c$ is an asymptote to the graph of $f(x)$.

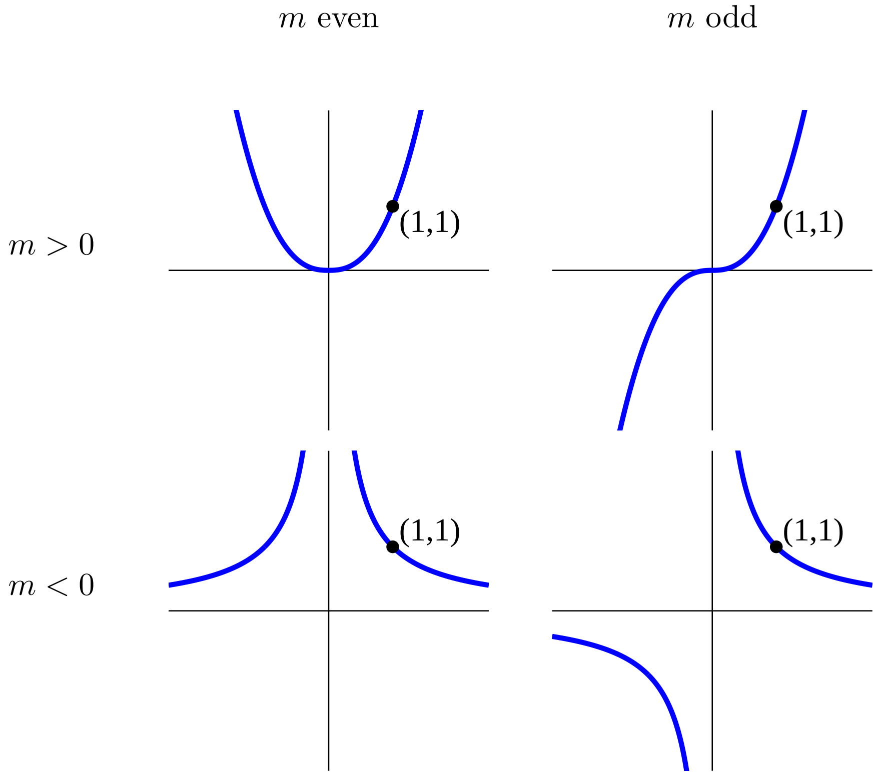


Some useful limit facts

Fact 2.3.r6. *Something goes to zero if and only if its reciprocal goes to $\pm\infty$.*

Fact 2.3.r7. *If m is a constant, then* $\lim_{x\to\infty} x^m = \begin{cases} \infty & \textit{if } m > 0, \textit{ and} \\ 0 & \textit{if } m < 0. \end{cases}$

For instance, here are the graphs of x^m for m any integer.



Fact 2.3.r8. *If $p(x)$ is a polynomial and $\ell t_p(x)$ is its lead term, then*

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} \ell t_p(x) \qquad \textit{and} \qquad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} \ell t_p(x).$$

2.3.r9. The lead term of $5x^3 - 2x^2 - 9x + 1$ is $5x^3$, so

$$\lim_{x\to\infty} (5x^3 - 2x^2 - 9x + 1) = \lim_{x\to\infty} 5x^3 = \infty, \text{ and}$$
$$\lim_{x\to-\infty} (5x^3 - 2x^2 - 9x + 1) = \lim_{x\to-\infty} 5x^3 = -\infty.$$

You can see the graphs of $5x^3 - 2x^2 - 9x + 1$ and $5x^3$ at

https://www.desmos.com/calculator/hadlqfy8no

2.3.r10. The lead term of $2x^4 + 3x^2 - 5$ is $2x^4$, so

$$\lim_{x\to\infty} (2x^4 + 3x^2 - 5) = \lim_{x\to\infty} 2x^4 = \infty, \text{ and}$$
$$\lim_{x\to-\infty} (2x^4 + 3x^2 - 5) = \lim_{x\to-\infty} 2x^4 = \infty.$$

Fact 2.3.r11. *If $p(x)$ and $q(x)$ are polynomials with lead terms $\ell t_p(x)$ and $\ell t_q(x)$, then*

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\ell t_p(x)}{\ell t_q(x)} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \frac{\ell t_p(x)}{\ell t_q(x)}.$$

2.3.r12. Use Fact 2.3.r11 to evaluate these limits:

a. $\displaystyle\lim_{x\to-\infty} \frac{x^2+1}{2x^2+x-4}$ b. $\displaystyle\lim_{x\to\infty} \frac{x+1}{2x^2+x-4}$ c. $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x-4}$ d. $\displaystyle\lim_{x\to-\infty} \frac{x^2+1}{3x-4}$

Solution:

a. $\displaystyle\lim_{x\to-\infty} \frac{x^2+1}{2x^2+x-4} = \lim_{x\to-\infty} \frac{x^2}{2x^2} = \lim_{x\to-\infty} \frac{1}{2} = \frac{1}{2}$. Consequently, $y = \frac{1}{2}$ is a horizontal asymptote to the graph of $y = \dfrac{x^2+1}{2x^2+x-4}$.

b. $\displaystyle\lim_{x\to\infty} \frac{x+1}{2x^2+x-4} = \lim_{x\to\infty} \frac{x}{2x^2} = \lim_{x\to\infty} \frac{1}{2x} = \text{“}\frac{1}{\infty}\text{”} = 0$. (The last “=” uses Fact 2.3.r6.) Therefore, $y = 0$ is a horizontal asymptote to the graph of $y = \dfrac{x+1}{2x^2+x-4}$.

c. $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x-4} = \lim_{x\to\infty} \frac{x^2}{3x} = \lim_{x\to\infty} \frac{x}{3} = \infty$.

d. $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x-4} = \lim_{x\to-\infty} \frac{x^2}{3x} = \lim_{x\to-\infty} \frac{x}{3} = -\infty$. Limits c and d together tell us that the graph of $\frac{x^2+1}{3x-4}$ has no horizontal asymptote.

end example 2.3.r12

Important: Facts 2.3.r7, 2.3.r8, and 2.3.r11 apply only when $x \to \pm\infty$. We can't use them when x goes to a number.

2.3.r13. Evaluate the limit.

a. $\displaystyle\lim_{x\to-\infty} (2x^3 - 4x^2 - 10^{23})$ b. $\displaystyle\lim_{x\to\infty} (-2x^4 + 3x + 1)$ c. $\displaystyle\lim_{x\to\infty} \frac{2x-1}{x+3}$

d. $\displaystyle\lim_{x\to-\infty} \frac{3x-1}{3-x^2}$ e. $\displaystyle\lim_{x\to\infty} \frac{2x^2+3x-1}{3-x}$ f. $\displaystyle\lim_{x\to-\infty} \frac{2x^2+3x-1}{3-x}$

Answers

2.3.r5a. 5/6 2.3.r5b. 1/4 2.3.r13a. “$\infty - 0$,” so the limit $= \infty$. 2.3.r13b. $= \lim_{x\to\infty} 4x^5 = \infty$. 2.3.r13c. $= \lim_{x\to\infty} \frac{2x}{x} = 2$. 2.3.r13d. $= \lim_{x\to\infty} \frac{3x}{-x^2} = \lim_{x\to\infty} \frac{3}{-x} = 0$. 2.3.r13e. $= \lim_{x\to\infty} \frac{2x^2}{-x}$ $= \lim_{x\to\infty} -2x = -\infty$. 2.3.r13f. $= \lim_{x\to-\infty} \frac{2x^2}{-x} = \lim_{x\to-\infty} -2x = \infty$.

2.4: Continuity

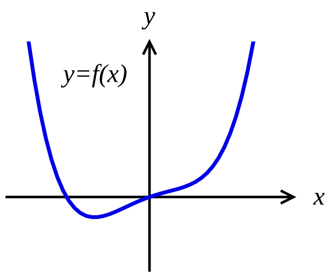
Definition 2.4.r1. *The function $f(x)$ is said to be* **continuous at a** *(an x-value) if*

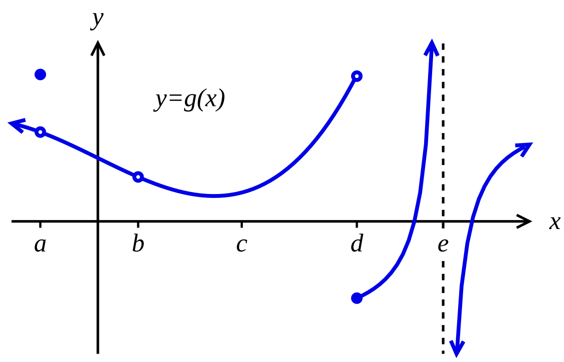
$$\lim_{x \to a} f(x) = f(a).$$

For f to be continuous at a means:

1. the limit $\lim_{x \to a} f(x)$ exists,
2. the function value $f(a)$ exists, and
3. the two are equal.

Informally, if you were drawing the graph of a function from left to right, the function is **discontinuous** (that is, not continuous) at the x-values where you have to lift your pencil off the paper.





2.4.r2. Where are the above functions $f(x)$ and $g(x)$ continuous? Discontinuous?

Solution:
a. f appears to be continuous everywhere.

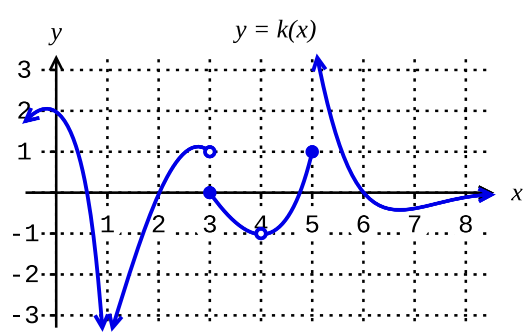
b. g is discontinuous at $x =$
a, where the limit value and function value exist but are unequal,
b, where the limit exists but function does not,
d, where the function exists but limit does not, and
e, where the neither limit nor function exist.

g appears to be continuous everywhere else.

end example 2.4.r2

2.4.r3. At what x-values, if any, is the function k pictured here discontinuous?

2.4.r4. At what x-values, if any, is the function p from 2.2.r3 discontinuous?



Definition 2.4.r5. *The function $f(x)$ is said to be* **continuous** *if it is continuous at every point a in its domain.*

Rules 10 and 11 from section 2.3 of these notes imply the following.

Fact 2.4.r6. *All polynomials are continuous on the entire real number line. All rational functions are continuous where they are defined.*

2.4.r7. Where is the given function continuous?

a. $3x^2 + x - 4x^7$ b. $\dfrac{x^2-1}{x^2-3x+2}$ c. $(2x+1)(x-4)$

Solution:
a. Since it is a polynomial, this function is continuous on $(-\infty, \infty)$.
b. This is a rational function. It's continuous everywhere that its denominator $x^2-3x+2 = (x-2)(x-1)$ is not zero, that is, all x other than 1 and 2. In interval form, the domain is $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$.
c. Don't let its factored form distract you. This function is a polynomial and is therefore continuous on $(-\infty, \infty)$.

end example 2.4.r7

2.4.r8. Where is the given function continuous?

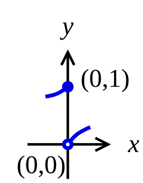
a. $x^2(x^3-8)$ b. $\dfrac{2x}{x^2-9}$ c. $\dfrac{2x+1}{x^2+9}$

A piecewise-defined function needn't be continuous at its "knots" (where its pieces join), even if it's made of continuous pieces.

2.4.r9. Where is the function $\alpha(x) = \begin{cases} x^2+1 & \text{if } x \le 0 \\ \frac{x}{x+1} & \text{if } x > 0 \end{cases}$ discontinuous?

Solution:

The polynomial x^2+1 is continuous for all real x, and the rational function $\frac{x}{x+1}$ is continuous for all $x \neq -1$. Therefore $\alpha(x)$ is continuous for all $x < 0$ (where it equals x^2+1) and all $x > 0$ (where it equals $\frac{x}{x+1}$).
The only possible discontinuity is at $x = 0$. It fact, α is discontinuous at $x = 0$. Because $0^2 + 1 = 1$ and $\frac{0}{0+1} = 0$, the two pieces of α don't meet continuously at $x = 0$. (See graph.)



end example 2.4.r9

2.4.r10. Where is the given function discontinuous?

a. $\beta(x) = \begin{cases} x^2 + 4x & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 < x < 1 \\ x^4 - 1 & \text{if } 1 \geq x \end{cases}$ b. $\gamma(x) = \begin{cases} \frac{1}{x-2} & \text{if } x < 1 \\ x^2 + 2x - 4 & \text{if } 1 \leq x \leq 2 \\ \frac{x}{x-1} & \text{if } 2 < x \end{cases}$

Answers

2.4.r3. Discontinuous at $x = 1$, 3, 4, and 5. 2.4.r4. 0, 1, 3. 2.4.r8a. $(-\infty, \infty)$.
2.4.r8b. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ 2.4.r8c. The denominator is never zero. Continuous on $(-\infty, \infty)$.
2.4.r10a. discontinuous only at $x = 0$. 2.4.r10b. discontinuous only at $x = 2$.

2.5: Average rate of change

If f is a function of x and $x_1 < x_2$ are two x-values, then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change** of f with respect to x between $x = x_1$ and $x = x_2$. The units of the rate of change are units of f per unit of x.

2.5.r1. If I start driving at time $x = 0$ and $f(x)$ equals the total distance I've driven at time x, then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{change in distance}}{\text{change in time}}$$

is my average velocity between times x_1 and x_2. If x is measured in hours and $f(x)$ is measured in miles, then my average velocity is measured in $\frac{\text{miles}}{\text{hour}}$, or miles per hour.

2.5.r2. If I own stock in a company and $g(x)$ is the value of my stock at time x, then

$$\frac{g(x_2) - g(x_1)}{x_2 - x_1} = \frac{\text{change in value}}{\text{change in time}}$$

is the average rate of change of my stock's value between times x_1 and x_2. If x is measured in years and $g(x)$ is measured in dollars, then this average rate of change is measured in $\frac{\text{dollars}}{\text{year}}$, or dollars per year.

2.5.r3. Find the average rate of change of the function between the given x-values.

a. $f(x) = (x-1)^2 + x + 1$; $x_1 = -1$, $x_2 = 1$

b. $g(x) = 4x - 9$; $x_1 = 2$, $x_2 = 5$

c. $k(x) = \dfrac{2}{x-3}$; $\quad x_1 = -2$, $x_2 = 1$

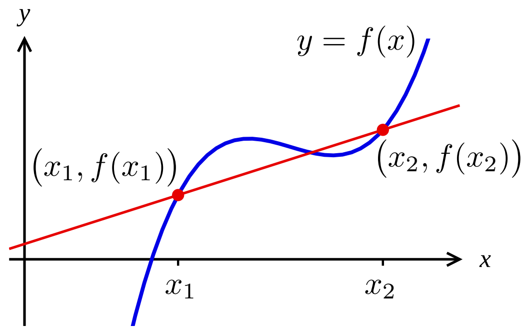
Any line that intersects a curve twice is called a **secant line**. The average rate of change

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the slope of the secant line joining the points

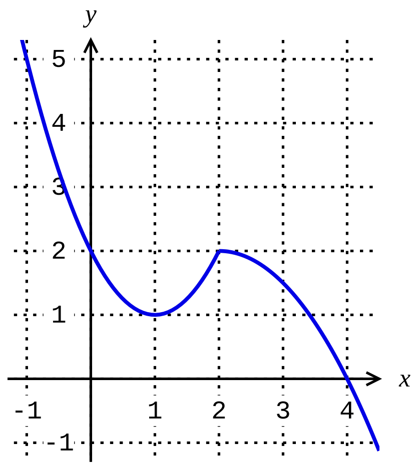
$$(x_1, f(x_1)) \quad \text{and} \quad (x_2, f(x_2))$$

on the graph of $f(x)$.



2.5.r4. Estimate the average rate of change of the function $g(x)$ graphed here between the two given x-values.

a. $x_1 = 1$, $x_2 = 3$ b. $x_1 = 0$, $x_2 = 4$ c. $x_1 = -1$, $x_2 = 2$



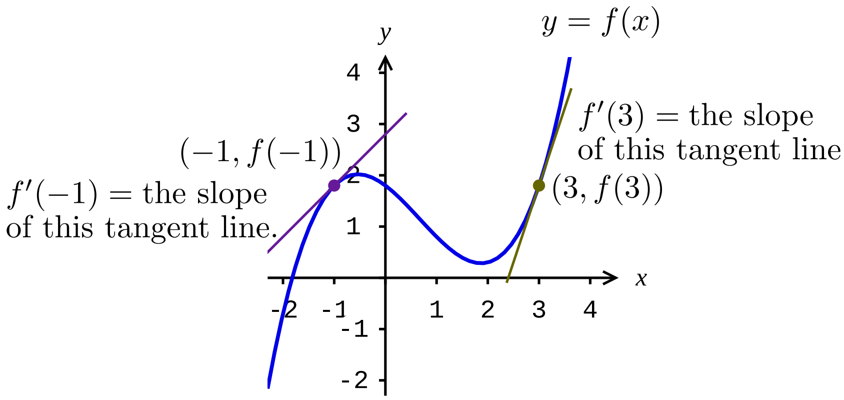
Answers

2.5.r3a. -1. 2.5.r3b. 4. 2.5.r3c. $-\frac{1}{5}$. 2.5.r4a. $\frac{1}{4}$. 2.5.r4b. $-\frac{1}{2}$. 2.5.r4c. -1.

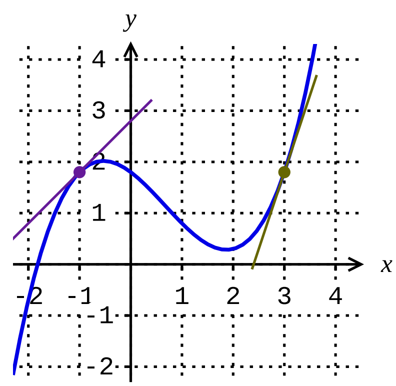
2.6: Instantaneous rate of change

The **derivative** of a function f is *another* function f' whose value at x is the slope of the line tangent to the graph of f at that x-value.

2.6.r1.



2.6.r2. Use the figure to approximate the values of $f'(-1)$, $f'(1)$, and $f'(3)$.



2.6.r3. If $g(x)$ is the funcion graphed in example 2.5.r4, estimate $g'(0)$, $g'(1)$, $g'(3)$, and $g(3)$.

The slope of a curve

We sometimes refer to $f'(x)$ as the slope of the curve $y = f(x)$ at the particular point $(x, f(x))$. That is, when we refer to the slope of a curve at a point, we mean the slope of the line tangent to the curve at that point.

2.6.r2, continued. Use the graph of $f(x)$ to approximate the x-values where ...

a. the slope of $y = f(x)$ is greatest.

b. the slope of $y = f(x)$ is least.

c. the slope of $y = f(x)$ is zero.

d. the slope of $y = f(x)$ is negative.

In section 2.5, we saw that the slope of a line secant to the graph of a function can be interpretted as an average rate of change of that function. In the same way, the slope of a line tangent to the graph of a function is interpretted as an **instantaneous** rate of change of that function, that is, its rate of change at one moment in time.

2.6.r4. I'm stopped at a red light. The light turns green at time $x = 0$, and I begin to accelerate down the road. If $f(x)$ equals the total distance I've driven at time x, then $f'(x)$ is my (instantaneous) velocity at time x. If we measure time x in seconds and distance $f(x)$ in meters, then $f'(x)$ is my velocity in meters per second. Assuming I keep accelerating for 10 seconds, which is greater, $f'(2)$ or $f'(4)$?

2.6.r5. A paper cup is filled with water. At time $x = 0$ a hole is punched in the bottom of the cup and water starts to leak out. If $g(x)$ is the volume in cm^3 of water remaining in the cup at time x in seconds, then $g'(x)$ is the rate at which water is leaking out of the cup at time x, measured in cm^3 per second. Is $g'(x)$ positive or negative? Hint: is g increasing or decreasing over time? Would you expect its tangent lines to be positively or negatively sloped?

2.6.r6. The cost of extracting x tons of ore from a copper mine is $C(x)$ dollars. What are the units of $C'(x)$? What does it mean to say that $C'(400) = \$20$?

2.6.r7. Suppose $P(x)$ is the cost in dollars of building a house of x square feet. What are the units of $P'(x)$? What interpretation can you give to the statement $P'(1800) = \$3500$?

2.6.r8. Suppose that the value of a piece of property is $v(x)$, measured in thousands of dollars, where x is time measure in years since the year 2020. What are the units of $v'(x)$? What does it mean to say that $v(3) = 25$ and $v'(3) = -3$?

Answers

2.6.r2. This problem requires that we estimate the slopes of three tanget lines. $f'(-1) \approx 1$, $f'(1) \approx -1$, and $f'(3) \approx 3$. 2.6.r3. The values of g' are slopes of tangent lines: $g'(0) \approx -2$, $g'(1) \approx 0$, and $g'(3) \approx -1$. $g(3)$ is the height of the curve $x = 3$, which we estimate to be $g(3) \approx 1.5$. 2.6.r3. a. $f'(x)$ appears to be greatest at $x \approx 3.5$ and at $x \approx -2.3$. b. $f'(x)$ is least at $x \approx 0.8$. c. $f'(x) = 0$ at about $x \approx -0.5$ and $x \approx 1.9$. d. $f'(x) < 0$ for for $-0.5 < x < 1.9$, approximately. 2.6.r4. My velocity increases as I accelerate, so my velocity at time $2 = f'(2) < f'(4) =$ my velocity at time 4. 2.6.r5. As water empties from the cup, $g(x)$ decreases. Its tangent lines have negative slope, so $g'(x)$ must be negative.
2.6.r6. $C'(x)$ is measured in dollars per ton. $C'(400) = \$20$ means that when I've already extracted 400 tons, the cost of extracting the next ton is approximately \$20. 2.6.r7. Units of $P'(x)$ are dollars per square foot. $P'(1800) = \$3500$ means that if you're planning to build a house of size 1800 square feet, the cost of adding one additional square foot to your plans is about \$3500. 2.6.r8. The units of $v'(x)$ are thousands of dollars per year. In 2023, the property was valued at \$25,000 and its value dropping at the rate of \$3,000 dollars per year.

2.7: Definition of the derivative and the power rule

In section 2.6, we introduced the idea of the derivative. In this section, we define it specifically as the limit of **difference quotients**.

Definition 2.7.r1. *The* **derivative** *of a function $f(x)$ is the limit*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

To **differentiate** is a funcion is to find its derivative. A function is **differentiable** where its derivative exists. (Since the derivative is a limit, and limits sometimes fail to exist, some functions fail to be differentiable.)
On exams and quizzes, you may be asked to differentiate a function using Definition 2.7.r1, as in this next example. See also Example 3, p. 163 of our text.

2.7.r2. Use the definition of derivative to differentiate $f(x) = \frac{2}{x}$:

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x}{x}\cdot\frac{2}{(x+h)} - \frac{2}{x}\cdot\frac{(x+h)}{(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{2x-2x-2h}{x(x+h)}\cdot\frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2h}{x(x+h)}\cdot\frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2}{x(x+h)} \\ &= \frac{-2}{x^2} \end{aligned}$$

end example 2.7.r2

2.7.r3. Use the definition of derivative to differentiate the given functions:

a. $f(x) = 3x+1$	b. $g(x) = 4x-7$	c. $k(x) = x^2 - 5x$
d. $\ell(x) = -2x^2 + x$	e. $m(x) = \dfrac{-1}{x}$	f. $n(x) = \dfrac{1}{2x}$

Differentiation rules

In the next few sections we learn some rules that allow us to find derivatives more easily than by using Definition 2.7.r1. These rules come in two types.
Below, $f(x)$ and $g(x)$ can be any differentiable functions, and c and n can be any constants.

I. A catalog of elementary functions and their derivatives.

If c and r are any constants, then

1. $(c)' = 0$
2. $(x^n)' = nx^{n-1}$ (the **power rule**)

II. Combination laws.

If f and g are differentiable functions, and if c is any constant, then

3. $\big(cf(x)\big)' = cf'(x)$
4. $\big(f(x) + g(x)\big)' = f'(x) + g'(x)$
5. $\big(f(x) - g(x)\big)' = f'(x) - g'(x)$

2.7.r4. Use the rules of differentiation to differentiate $f(x) = \frac{1}{2}x^2 - 5\sqrt{x} + \frac{1}{x^2}$.

Solution: First write the function in terms of power functions.

$$f(x) = \frac{1}{2}x^2 - 5x^{1/2} + x^{-2}$$

Now apply rule 4:

$$f'(x) = \left(\frac{1}{2}x^2 - 5x^{1/2} + x^{-2}\right)' = \left(\frac{1}{2}x^2\right)' - (5x^{1/2})' + (x^{-2})'$$

Now rule 3:

$$= \frac{1}{2}(x^2)' - 5(x^{1/2})' + (x^{-2})'$$

Finally, apply the power rule three times and simplify:

$$= \frac{1}{2}(2x^1) - 5\left(\frac{1}{2}x^{-1/2}\right) - 2x^{-3} = x - \frac{5}{2}x^{-1/2} - 2x^{-3}.$$

end example 2.7.r4

Warning: the value of the derivative of a function is different from the derivative of the value of the function. The latter is always 0.

2.7.r5. Find the derivative of the given function:

a. $x^4 + 5x - 3$ b. $\sqrt{x} + \sqrt{\dfrac{9}{x}}$ c. $\sqrt[4]{x} - 3x$

d. $\dfrac{4}{x^3}$ e. πr^2 f. $\dfrac{4}{3}\pi r^3$

g. $-x^3 + 3^4$ h. $4x^{1.2} - 2e^{1.2}$ i. $\sqrt{2} + 3\sqrt[3]{x^4}$

Answers

2.7.r3a. $f'(x) = 3$ 2.7.r3b. $g'(x) = 4$ 2.7.r3c. $k'(x) = 2x - 5$ 2.7.r3d. $\ell'(x) = -4x + 1$
2.7.r3e. $m'(x) = \frac{1}{x^2}$ 2.7.r3f. $n'(x) = \frac{-1}{2x^2}$ 2.7.r5a. $4x^3 + 5$ 2.7.r5b. $\frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2}$
2.7.r5c. $\frac{1}{4}x^{-3/4} - 3$ 2.7.r5d. $\frac{-12}{x^4}$ 2.7.r5e. $2\pi r$ 2.7.r5f. $4\pi r^2$ 2.7.r5g. $-3x^2$ 2.7.r5h. $4.8x^{0.2}$
2.7.r5i. $4x^{1/3}$

2.8: More about the derivative

The derivative is both a slope and a rate of change.

2.8.r1. Find an equation of the line tangent to $y = x^{3/2} - 6x^{1/2} - 1$ at $x = 9$.

Solution: The derivative of y is $y' = \frac{3}{2}x^{1/2} - 3x^{-1/2}$. Calculate y and y' at $x = 9$:

$$\begin{aligned} y &= 9^{3/2} - 6 \cdot 9^{1/2} - 1 \\ &= (\sqrt{9})^3 - 6\sqrt{9} - 1 \\ &= 3^3 - 6 \cdot 3 - 1 = 8 \\ y' &= \frac{3}{2} \cdot 9^{1/2} - 3 \cdot 9^{-1/2} \\ &= \frac{3}{2}\sqrt{9} - \frac{3}{\sqrt{9}} \\ &= \frac{9}{2} - 1 = \frac{7}{2} \end{aligned}$$

The tangent line touches the curve at $x = 9$, so the point $(9, 8)$ is on the line. Its slope is $\frac{7}{2}$, so the line is given by the equation

$$y - 8 = \frac{7}{2}(x - 9).$$

2.8.r1, continued. Suppose an object moving along the number line is at position $s = t^{3/2} - 6t^{1/2} - 1$ (meters) at time t (seconds). The derivative

$$\frac{ds}{dt} = \frac{3}{2}t^{1/2} - 3t^{-1/2} \quad \text{(meters/second)}$$

is the instantaneous velocity of the object at time t. For instance, at $t = 9$, when the object is at position 8 meters, its velocity is $\frac{7}{2}$ meters per second.

end example 2.8.r1

2.8.r2. Find an equation of the line tangent to the graph of $f(x)$ at the given x-value. Use a graphing device to plot the curve and your equation. Does it appear that you have the right equation?

a. $4x^2 - 5x - 9;\ x = 1$ b. $5x + 1;\ x = 4$ c. $2x^{3/2} - 5x^2 + 1;\ x = 1$

d. $x^2(x - 1);\ x = 2$ e. $2 + \frac{1}{x};\ x = \frac{1}{3}$ f. $4x - \frac{5}{x^2};\ x = \frac{1}{2}$

Answers

2.8.r2a. $y + 10 = 3(x - 1)$ 2.8.r2b. $y = 5x + 1$ 2.8.r2c. $y + 2 = -7(x - 1)$ 2.8.r2d. $y - 4 = 8(x - 2)$ 2.8.r2e. $y = -9x + 8$ 2.8.r2f. $y + 18 = 84(x - \frac{1}{2})$

2.9: Marginal analysis

Here are some of the economic functions introduced in section 1.3:

x : The number of units sold.

$D(x)$: The **demand** function, that is, the highest price at which buyers in the market are willing to buy x units.

$C(x)$: The merchant's **cost** of obtaining x units of their product.

$R(x) = xD(x)$: The merchant's **revenue** when they sell x units.

$P(x) = R(x) - C(x)$: The merchant's **profit** when they obtain and sell x units.

New in this section:

$\overline{C}(x) = \dfrac{C(x)}{x}$: The merchant's **average cost** per unit of obtaining x units.

Generally, in economics, "marginal" means "the derivative of."

$R'(x)$: **Marginal revenue**.

$C'(x)$: **Marginal cost**.

$\overline{C}'(x)$: **Marginal average cost**.

$P'(x)$: **Marginal profit**. Because profit is revenue minus cost

$$P(x) = R(x) - C(x),$$

marginal profit is maginal revenue minus marginal cost.

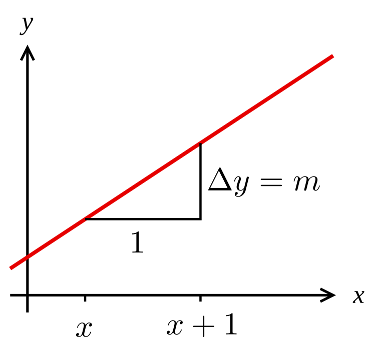
$$P'(x) = R'(x) - C'(x)$$

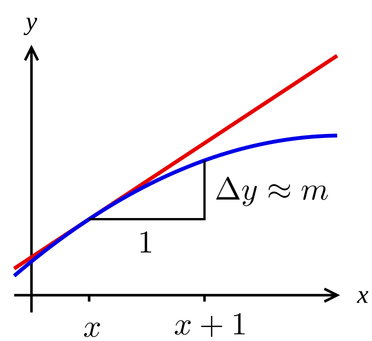
The meaning of "marginal"

Because its slope is constant, the slope of a line is the change in y along the line corresponding to a 1-unit increase in x at any x along the line.

$$m = \frac{\Delta y}{\Delta x} = \Delta y \text{ if } \Delta x = 1.$$

See below left.





Since its slope is nonconstant, the slope of a curve at a point is *approximately* the change in y along the curve corresponding to a 1-unit increase in x from that point. See above right.
Consequently, marginal cost $C'(x)$ is approximately the change in cost $C(x)$ due to a one-unit increase from x to $x+1$. (In that sense, it is approximately the cost of the $x+1$ unit.) Marginal profit is approximately the change in profit due to a one-unit increase in x, etc.. In general,

Marginal [**word**] is the approximate change in [**word**] due to a one-unit increase in x.

2.9.r1. Suppose it costs a sock manufacturer in Bangladesh $C(x) = 10,000 + 75x$ taka* to make x socks in a day. What is the manufacturer's fixed cost per day? What is their marginal cost? Find the manufacturer's average cost and marginal average cost at $x = 10,000$.

Solution: Fixed cost is $C(0) = 10,000$ Tk. Marginal cost is the derivative of cost, that is, $C'(x) = 75$ (which in this example is **exactly** the cost of making one more sock). Average cost is $\overline{C}(x) = (10,000 + 75x)/x = 10,000x^{-1} + 75$. At $x = 10,000$, average cost is $\frac{10,000}{10,000} + 75 = 76$ taka per sock. Marginal average cost is $\overline{C}'(x) = -10,000x^{-2}$, and $\overline{C}'(10,000) = -\frac{10,000}{(10,000)^2} = -10^4/10^8 = -10^{-4} = -0.0001$ This means, for instance, that when the manufacturer makes 10,000 socks per day, increasing output by 1 sock per day causes their average cost per sock to go down a very small amount.

2.9.r2. Suppose cost $C(x) = 100 + 50x + \frac{1}{20}x^2$ and revenue $R(x) = 100x$. Find average cost, profit and marginal profit.

2.9.r3. Suppose *average* cost is $\overline{C}(x) = 500x^{-1} + 20 + \frac{1}{50}x$. Find cost $C(x)$ and marginal cost.

Answers

2.9.r2. Average cost $= \overline{C} = C(x)/x = 100x^{-1} + 50$. Profit $= P(x) = R(x) - C(x) = 50x - 100 - \frac{1}{20}x^2$. Marginal profit $= P'(x) = 50 - \frac{1}{10}x$. 2.9.r3. $C(x) = 500 + 20x + \frac{1}{50}x^2$. Marginal cost is the derivative: $C'(x) = 20 + \frac{1}{25}x$.

* The taka is the the currency of Bangladesh.

3.1: The Product and Quotient Rules

II. Combination laws, continued.

If f and g are differentiable functions, then

6. $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$ (the **product rule**)

7. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$, provided $g(x) \neq 0$. (the **quotient rule**)

Note that $(fg)' \neq f'g'$ and $(f \div g)' \neq f' \div g'$

Always simplify at every step to make your next step easier and more likely to be correct.

3.1.r1. Find the derivative of the following functions.

a. $(3x^2 + 5x)(2x + 1)$ b. $\dfrac{3x^2 + 5x}{2x + 1}$

Solution: a. As an exercise, we'll differentiate two ways and compare our answers. First, by the product rule,

$$\begin{aligned}\big((3x^2+5x)(2x+1)\big)' &= (3x^2+5x)'(2x+1) + (3x^2+5x)(2x+1)' \\ &= (6x+5)(2x+1) + (3x^2+5x)2 \\ &= (12x^2+16x+5) + (6x^2+10x) = 18x^2+26x+5\end{aligned}$$

Again, this time expanding the product before differentiation.

$$\big((3x^2+5x)(2x+1)\big)' = (6x^3+13x^2+5x)' = 18x^2+26x+5.$$

b. By the quotient rule,

$$\begin{aligned}\left(\frac{3x^2+5x}{2x+1}\right)' &= \frac{(3x^2+5x)'(2x+1) - (3x^2+5x)(2x+1)'}{(2x+1)^2} \\ &= \frac{(6x+5)(2x+1) - (3x^2+5x)2}{(2x+1)^2} \\ &= \frac{(12x^2+16x+5) - (6x^2+10x)}{(2x+1)^2} \\ &= \frac{6x^2+6x+5}{(2x+1)^2}\end{aligned}$$

end example 3.1.r1

3.1.r2. Find the derivative of the given function.

a. $(3x+1)(2x-3)$ b. $\dfrac{x}{3x^2+1}$ c. $\dfrac{3x^3+2}{x^2}$

d. $\dfrac{x}{x+\frac{2}{x}}$ e. $(r-\sqrt{r})(r+\sqrt{r})$ f. $\dfrac{x^2+4}{x^2-4}$

3.1.r3. Find the equation of the line tangent to $y = \dfrac{x}{3x^2+1}$ at $x = 1$.

3.1.r4. Use these values of $f(x)$ and $g(x)$ and their derivatives at $x = 1$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	3	4	-5	-6

to evaluate the **derivatives** of the following functions at $x = 2$:

a. $f(x)g(x)$ b. $g(x)f(x)$
c. $xg(x)$ d. $(x^2-1)f(x))$

Answers

3.1.r2a. $12x-7$ 3.1.r2b. $\frac{1-3x^2}{(3x^2+1)^2}$ 3.1.r2c. $3-4x^{-3}$ 3.1.r2d. $\frac{4x^{-1}}{(x+2x^{-1})^2}$ 3.1.r2e. $2r-1$
3.1.r2f. $\frac{-16x}{(x^2-4)^2}$ 3.1.r3. $y-\frac{1}{4} = -\frac{1}{8}(x-1)$ 3.1.r4a. $f'(2)g(2)+f(2)g'(2) = 4(-5)+3(-6) = -38$. 3.1.r4b. (same as a.) 3.1.r4c. The derivative is $g(x)+xg'(x)$. At $x = 2$, this is $g(2)+2g'(2) = -5+2(-6) = -17$ 3.1.r4d. The derivative is $2xf(x)+(x^2-1)f'(x)$. At $x = 2$, this is $2\cdot 2f(2)+(2^2-1)f'(2) = 12+3\cdot(4) = 24$

3.2: The Chain Rule

II. Combination laws (continued).

If f and g are differentiable functions, then

8. $\big(f\big(g(x)\big)\big)' = f'\big(g(x)\big)g'(x)$ (the **chain rule**)

Here's another way to picture the chain rule. If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The most common way we see the chain rule in this course is when the last operation is a power function. Here's what that looks like.

The generalized power rule 3.2.r1. $\Big(\big(g(x)\big)^n\Big)' = n\big(g(x)\big)^{n-1}g'(x).$

3.2.r2. Find the derivative of the given function.

a. $\sqrt{x^2 - 5x}$ b. $(2x-1)^3(x^3-4x)$ c. $(2x-1)^3(x^3-4x)^4$

d. $\sqrt{\dfrac{x+1}{x-1}}$ e. $(x - 5\sqrt{x})^2$ f. $x\sqrt{x^3-1}$

3.2.r3. Use these values of $f(x)$ and $g(x)$ and their derivatives at $x = 1$, 2, and 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	-3	2	6
2	1	-4	3	7
3	2	-5	1	4

to evaluate the **derivatives** of the following functions at $x = 1$:

a. $g(f(x))$ b. $f(g(x))$

c. $f(f(x))$ d. $\big(g(x)\big)^2$

Answers

3.2.r2a. $\frac{2x-5}{2\sqrt{x^2-5x}}$ 3.2.r2b. $6(2x-1)^2(x^3-4x)+4(2x-1)^3(3x^2-4)$, or $(2x-1)^2\Big(6(x^3-4x)+4(2x-1)(3x^2-4)\Big)$
3.2.r2c. $6(2x-1)^2(x^3-4x)^4 + 4(2x-1)^3(x^3-4x)^3(3x^2-4)$, or
$(2x-1)^2(x^3-4x)^3\Big(6(x^3-4x)+4(2x-1)(3x^2-4)\Big)$ 3.2.r2d. $\left(\frac{-1}{(x-1)^2}\right)\left(\frac{x-1}{x+1}\right)^{1/2}$
3.2.r2e. $2(x-5\sqrt{x})(1-\frac{5}{2}x^{-1/2})$ 3.2.r2f. $\sqrt{x^3-1}+\frac{1}{2}\frac{3x^3}{\sqrt{x^3-1}}$ 3.2.r3a. $g'(f(1))f'(1) = g'(3)f'(1) = 4(-3) = -12$ 3.2.r3b. $f'(g(1))g'(1) = f'(2)g'(1) = -4\cdot 6 = -24$ 3.2.r3c. $f'(f(1))f'(1) = f'(3)f'(1) = (-5)(-3) = 15$
3.2.r3d. $2g(1)g'(1) = 2\cdot 2\cdot 6 = 24$

3.4: Increasing and decreasing functions

Definition 3.4.r1. *A function $f(x)$ is* **increasing** *whereever*

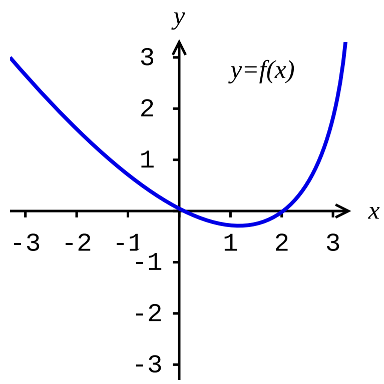
$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2)$$

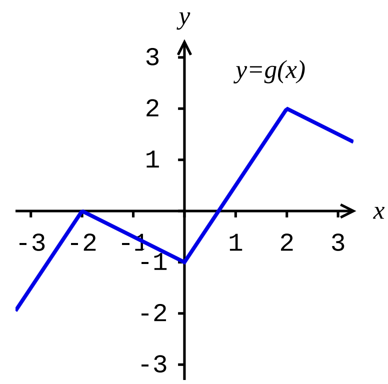
. $f(x)$ is **decreasing** *whereever*

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2)$$

.

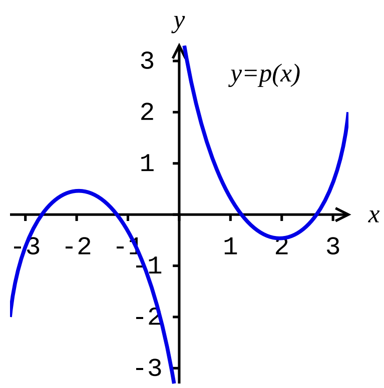
A function is increasing whereever its graph *rises* as it moves to the right, and it decreases whereever its graph *falls* as it moves to the right.

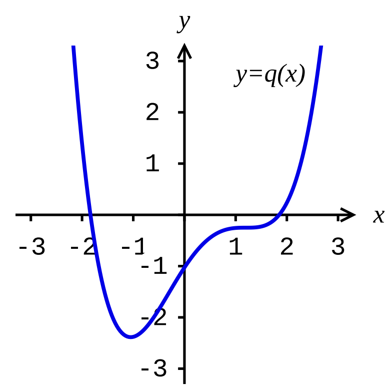




3.4.r2. See the graphs above. On what open intervals (in their domains) are f and g increasing? On what open intervals are they decreasing?

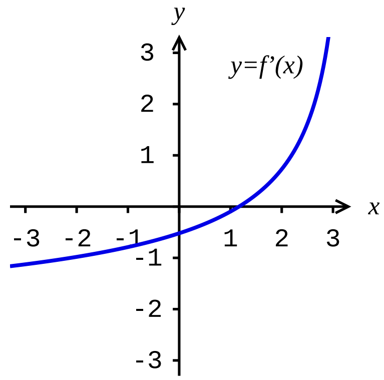
Solution: Any graph is open to some interpretation, but it appears that $f(x)$ is increasing on the interval $(1.2, \infty)$ and decreasing on $(-\infty, 1.2)$. The function $g(x)$ is increasing on the intervals $(-3.2, -2)$ and $(0, 2)$ and decreasing on the intervals $(-2, 0)$ and $(2, 3.2)$.

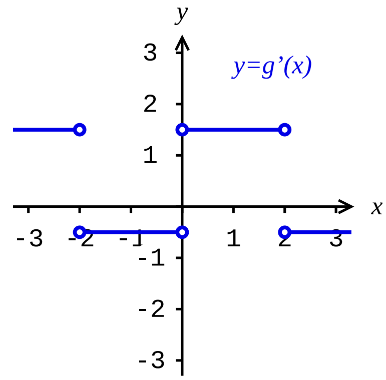




3.4.r3. See the graphs above. On what open intervals are p and q increasing? On what open intervals are they decreasing?

Fact 3.4.r4. *Assuming it is differentiable, a function is increasing where its derivative is positive, and decreasing where its derivative is negative.*





3.4.r2, continued. The derivatives of f and g are shown here. Notice that when f is increasing, f' is positive. In other words, when the graph of f is rising, the graph of f' is above the x-axis. Likewise, when f is descreasing, f' is negative. That is, when the graph of f is falling, the graph of f' is below the x-axis.

The graph of g consists of line segments of constant slope, so the graph of g' consists of horizonal lines of constant altitude. On the intervals where g is increasing, g' is positive. On the intervals where g is decreasing, g' is negative.

3.4.r2, continued. Calculate the slopes of the line segments of g. What are the altitudes of the graph of g'?

3.4.r5. Make a sign chart of the derivative to determine where the function is increasing and where it is decreasing.

a. $v(x) = 2x^3 + \frac{1}{2}x^2 - x - 2$ b. $w(x) = \dfrac{x}{x-2}$

Solution:

a. Differentiate to find $v'(x) = 6x^2 + x - 1$. To find where v' is positive and negative, first find where it is zero or undefined:

$$v'(x) = 6x^2 + x - 1 = (3x-1)(2x+1) = 0 \text{ at } x = \frac{1}{3} \text{ and} x = -\frac{1}{2}.$$

Since $v'(x)$ is a polynomial, it is defined for all real numbers x.

Here's what we know so far about the signs of v':

$v'(x)$:	0	0
x :	$-\frac{1}{2}$	$\frac{1}{3}$

To complete the sign chart, evaluate $v'(x)$ at sample points chosen to the left of $-\frac{1}{2}$, between $-\frac{1}{2}$ and $\frac{1}{3}$, and to the right of $\frac{1}{3}$.

x	-1	0	1
$v'(x)$	8	-1	6

Conclusion:

$v'(x):$	$++++++++++0---------------0+++++++$	
$x:$	$-\frac{1}{2}$	$\frac{1}{3}$

$v(x)$ is increasing on the intervals where $v'(x) > 0$, that is, $(-\infty, -\frac{1}{2})$ and $(\frac{1}{3}, \infty)$.
$v(x)$ is decreasing where $v'(x) < 0$, that is, on the interval $(-\frac{1}{2}, \frac{1}{3})$.

b. Use the quotient rule to find

$$w'(x) = \left(\frac{x}{x-2}\right)' = \frac{x'(x-2) - x(x-2)'}{(x-2)^2} = \frac{-2}{(x-2)^2}$$

w' will change signs only where it equals zero or does not exist. $w'(x)$ does not exist at $x = 2$, where its denominator is zero. $w'(x)$ never equals zero because is numerator is never 0. (For a fraction to be zero, its numerator must be zero.) The only place $w'(x)$ might change sign is at $x = 2$:

$w'(x):$	DNE
$x:$	2

When we compute $w'(x)$ for any x not equal 2, the result is $\frac{\text{negative}}{\text{positive}}$ = negative. Therefore, the sign chart is:

$w'(x):$	$------$DNE$------$
$x:$	2

Conclusion: $w(x)$ is decreasing on $(-\infty, 2)$ and on $(2, \infty)$.
See the graphs of v and w at https://www.desmos.com/calculator/gwdna13sei.

3.4.r6. Make a sign chart of the derivative to determine where the function is increasing and where it is decreasing.

a. $-(4x-5)^2$ b. $x^3 + x^2 - 5x - 5$ c. $\dfrac{x^3+54}{x}$

Answers

3.4.r3. p increases on $(-\infty, -2)$ and $(2, \infty)$ and decreases on $(-2, 0)$ and on $(0, 2)$. q increases on $(-1, \infty)$ and decreases on $(-\infty, -1)$. 3.4.r4. The line segments of the graph of g have slopes $\frac{3}{2}$ and $-\frac{1}{2}$, so the graph of g' consists of horizontal lines are altitude $\frac{3}{2}$ and $-\frac{1}{2}$. 3.4.r6a. inc on $(-\infty, \frac{5}{4})$; dec on $(\frac{5}{4}, \infty)$. 3.4.r6b. inc on $(-\infty, -\frac{5}{3})$ and on $(1, \infty)$; dec on $(-\frac{5}{3}, 1)$. 3.4.r6c. inc on $(3, \infty)$; dec on $(-\infty, 0)$ and on $(0, 3)$.

3.5: Local extrema, critical values, and the first derivative test

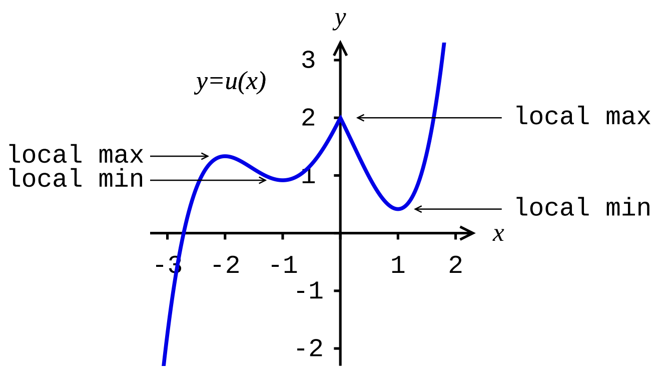
Vocabulary:

maximum (*pl* **maxima**): a function's greatest value
minimum (*pl* **minima**): a function's least value
extremum (*pl* **extrema**): a maximum or minimum

We study two types of extrema in calculus: **local** and **absolute**.

Local exrema

The local extrema of a function are the y-values at its peaks and valleys.



3.5.r1. The function u graphed above as four local extrema:

a local maximum at $x = -2$,
a local minimum at $x = -1$,
a local maximum at $x = 0$, and
a local minimum at $x = 1$.

The extrema themselves are the values of $u(x)$ at those points. We can approximate these from the graph:

$u(-2) \approx 1.3$ is a local maximum,
$u(-1) \approx 1$ is a local minimum,
$u(0) \approx 2$ is a local maximum, and
$u(1) \approx 0.5$ is a local minimum.

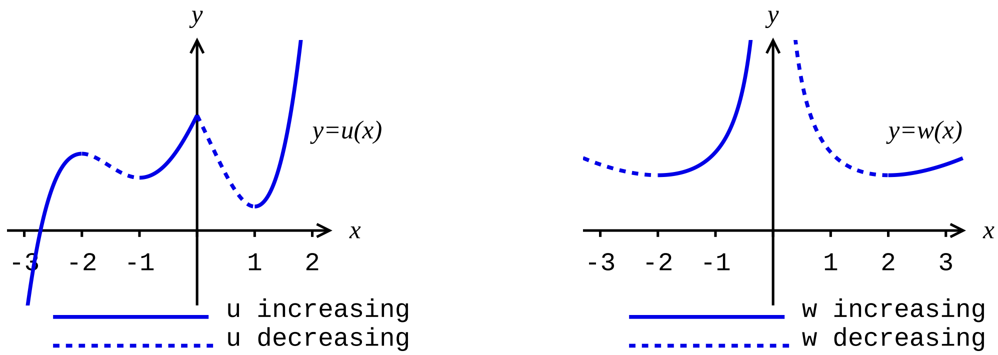
Critical points and the first derivtive test

Local extrema occur where the funcion exists and changes from increasing to decreasing or decreasing to increasing. At such points, if the derivative exists, it must equal zero.

Definition 3.5.r2. *An* **critical number** *or* **critical point** *of a function is a point in the function's domain where the derivative is either zero or nonexistant. A funtion's local extrema can only occur at its critical points.*

3.5.r1, continued. The function u graphed below left has
a local maximum at $x = -2$, where u changes from increasing to decreasing,
a local minimum at $x = -1$, where u changes from decreasing to increasing,
a local maximum at $x = 0$, where u changes from increasing to decreasing, and
a local minimum at $x = 1$, where u changes from decreasing to increasing.

$x = -2$, -1, 0, and 1 are critical points for u because
$u'(-2) = u'(-1) = u'(1) = 0$, and
$u(0)$ exists but $u'(0)$ does not, due to the corner at $x = 0$.

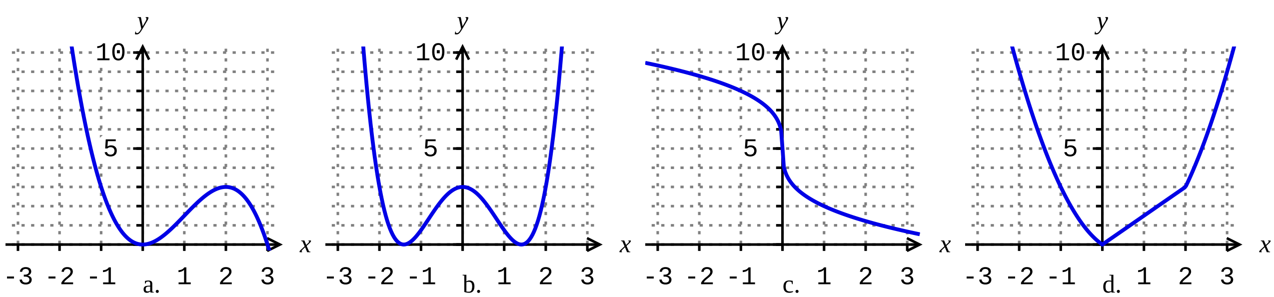


3.5.r3. The function w graphed above right has
a local minimum at $x = -2$, where w changes from decreasing to increasing, and
a local minimum at $x = 2$, where w changes from decreasing to increasing.

w changes from increasing to decreasing on at $x = 0$ but since $w(0)$ does not exist, there's no local max at $x = 0$.

$x = -2$ and 2 are critical points for w because $w'(-2) = w'(2) = 0$.

3.5.r4. Find the critical points of each of the functions graphed below. At which of these critical points does the function have a local extremum?



1st Derivative Test for Local Extrema 3.5.r5. *Suppose c is a critical point of f. If f' changes sign at c:*

$$\begin{array}{rc} f'(x): & - - - - \; + + + + \\ \hline x: & c \end{array} \quad \Longrightarrow \quad f \textit{ has a local minimum at } c.$$

$$\begin{array}{rc} f'(x): & + + + + \; - - - - \\ \hline x: & c \end{array} \quad \Longrightarrow \quad f \textit{ has a local maximum at } c.$$

If f' does not change sign at c, then f does not have a local extremum at c.

3.4.r5, continued. Apply the first derivative test to the sign charts for v and w found in the section 3.4 and conclude that v has a local maximum at $x = -\frac{1}{2}$ and a local minimum at $x = \frac{1}{3}$. Since w has no critical points, it has no local extrema. See their graphs at https://www.desmos.com/calculator/gwdna13sei.

3.5.r6. Find where $s(x) = x^4 - 18x^2 + 1$ has its local extrema using the first derivative test.

Solution: First find the critical points. $s'(x) = 4x^3 - 36x$ is defined for all real x so the only critical points are the solutions to $s'(x) = 0$. Solve by factoring:

$$0 = 4x^3 - 36x = 4x(x^2 - 9) = 4x(x-3)(x+3) \quad \Longrightarrow \quad x = -3,\ 0,\ \text{or } 3$$

Our sign chart for s' so far looks like this:

$$\begin{array}{rccc} s'(x): & 0 & 0 & 0 \\ \hline x: & -3 & 0 & 3 \end{array}$$

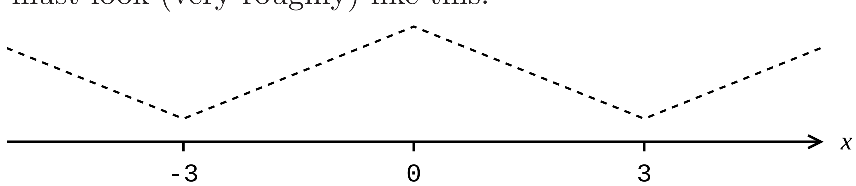
To complete the sign chart, determine the sign of $s'(x)$ at sample points chosen to the left of -3, between -3 and 0, between 0 and 3, and to the right of 3. Tip: it's easiest to use the factorization of s' found above.

x	-4	-1	1	4
$s'(x)$	$4(-4)(-7)(-1)$ < 0	$4(-1)(-4)(2)$ > 0	$4(1)(-2)(4)$ < 0	$4(4)(1)(7)$ > 0

Now we know the sign chart completely:

$$\begin{array}{rc} s'(x): & - - - - \, 0 + + + + + + \, 0 - - - - - - - \, 0 + + + + + \\ \hline x: & \quad -3 \qquad\qquad 0 \qquad\qquad 3 \end{array}$$

The graph of $s(x)$ must look (very roughly) like this:



Therefore, s has local minima at $x = -3$ and $x = 3$ and a local maximum at $x = 0$.

end example 3.5.r6

3.4.r6, continued. Where do these functions have local extrema?

a. $-(4x-5)^2$ b. $x^3 + x^2 - 5x - 5$ c. $\dfrac{x^3+54}{x}$

3.5.r7. Where do these functions have local extrema?

a. $x^2 - \dfrac{1}{3}x^3$ b. $3x^2 - 5x + 2$ c. $\dfrac{x^2+9}{x}$ d. $\dfrac{x^2-4}{x}$

Answers

3.5.r4. a. Critical pts: $x = 0$ and 2, where the derivative equals zero. Local min at $x = 0$. Local max at $x = 2$. b. Critical pts: $x = -1.5$, 0, and 1.5, where the derivative equals zero. Local min at $x = -1.5$ and 1.5. Local max at $x = 0$. c. Critical pts: $x = 0$, where the derivative does not exist. No local extremum at $x = 0$. d. Critical pts: $x = 0$ and 2, where the derivative does not exist. Local min at $x = 0$. No local extremum at $x = 2$. 3.5.r6a. loc max at $x = \frac{5}{4}$; no loc min. 3.5.r6b. loc max at $x = -\frac{5}{3}$; loc min at $x = 1$. 3.5.r6c. loc min at $x = 3$; no loc max. 3.5.r7a. loc max at $x = 2$; loc min at $x = 0$.
3.5.r7b. loc min at $x = \frac{5}{6}$; no loc max. 3.5.r7c. loc max at $x = -3$; loc min at $x = 3$. 3.5.r7d. no loc extrema.

3.6: Absolute extrema

The **absolute maximum** of a function on an interval is the greatest value of the function on that interval. Similarly, the **absolute minimum** of a function on an interval is its least value on that interval.
The absolute extrema of a function over an interval can change with the interval, as the next example shows.

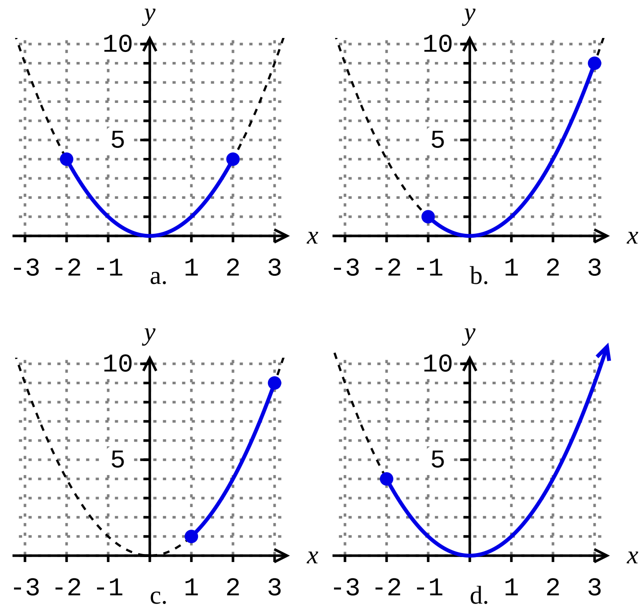
3.6.r1. Here are the absolute maximum and minimum of x^2 on various intervals:

a. The absolute max and min of x^2 on $[-2, 2]$ are the greatest and least values of y along the curve $y = x^2$ for $-2 \le x \le 2$. On $[-2, 2]$, the absolute max of x^2 is 4 and the absolute min is 0.

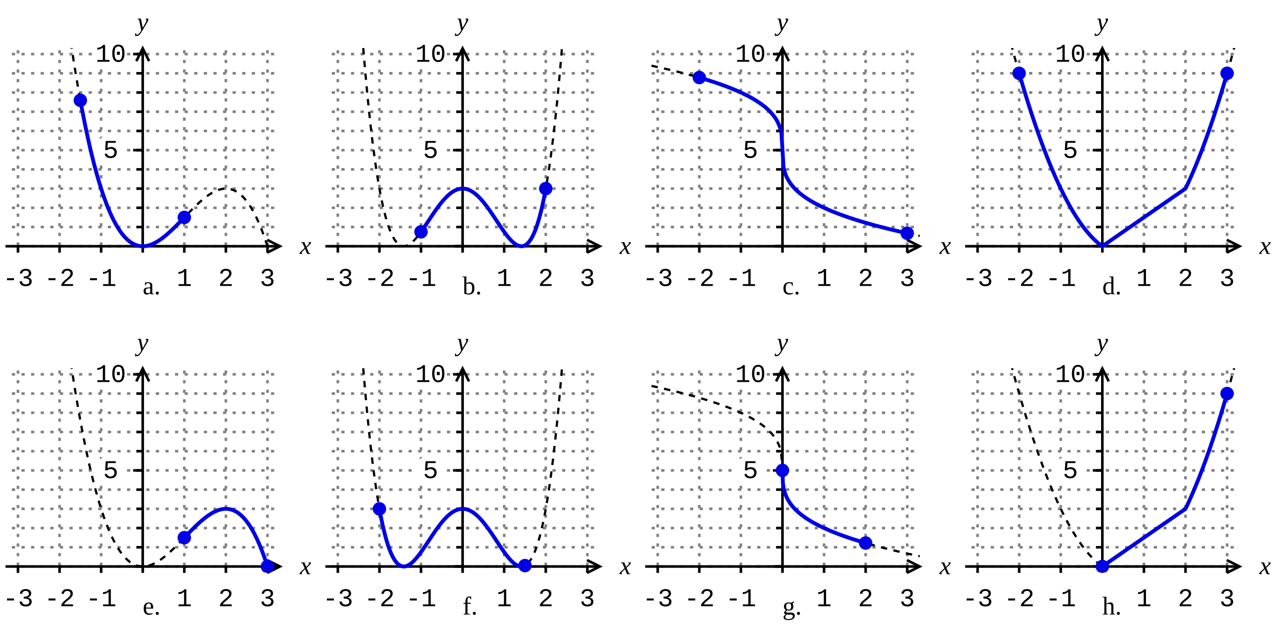
b. On $[-1, 3]$, the absolute max of x^2 is 9 and the absolute min is 0.

c. On $[1, 3]$, the absolute max of x^2 is 9 and the absolute min is 1.

d. On $[-2, \infty)$, x^2 has no absolute max, but its absolute min is 0.



3.6.r2. Find the absolute extrema of the function on the closed interval shown in each graph. State the x-value where each occurs.



Examples above illustrate the next fact.

Fact 3.6.r3. *The absolute extrema of a function on a closed interval will occur either at the endpoints of the interval or at critical points of the function* **inside** *the interval.*

3.6.r4. Find the absolute extrema of $s(x) = x^4 - 18x^2 + 1$ on the interval $[-1, 4]$.

Solution: We discovered in example 3.5.r6 that the critical points of $s(x)$ are $x = -3$, 0, and 3. The first of these is outside $[-1, 4]$, so the absolute extrema of s on this interval must be among its values at these four points:

x	-1	0	3	4
$s(x)$	-16	1	-80	-31

Therefore, the absolute max must be 1 and the absolute min is -80.

end example 3.6.r4

3.6.r5. Find the absolute extrema of the function on the given interval.

a. $-(7x-4)^2$ on $[-1, 1]$

b. $-(7x-4)^2$ on $[0, 2]$

c. $x^3 + 3x^2 - 45x$ on $[3, 5]$

d. $x^3 + 3x^2 - 45x$ on $[-2, 4]$

e. $\dfrac{x^3 - 16}{x}$ on $[-3, -1]$

f. $\dfrac{x^3 - 16}{x}$ on $[1, 3]$

g. $\dfrac{x^2 + 9}{x}$ on $[1, 4]$

h. $3x^2 - x^3$ on $[-1, 3]$

Answers

3.6.r2. a. Abs max is about 7.5, at $x = -1.5$. Abs min is 0 at $x = 0$. b. Abs max is 3, occuring at both $x = 0$ and $x = 2$. Abs min is 0 at $x = 0$. c. Abs max is about 8.9, at $x = -2$. Abs min is about 0.75 at $x = 3$. d. Abs max is 9, occuring at both $x = 0$ and $x = 2$. Abs min is 0 at $x = 0$. e. Abs max is 3, at both $x = 2$. Abs min is 0 at $x = 3$. f. Abs max is 3, at $x = -2$ and $x = 0$. Abs min is 0 at $x = -1.5$ and $x = 1.5$. g. Abs max is 5, at $x = 0$. Abs min is about 1 at $x = 2$. h. Abs max is 9, at $x = 3$. Abs min is 0 at $x = 0$. 3.6.r5a. Abs max $= 0$ at $x = \frac{4}{7}$. Abs min $= -121$ at $x = -1$. 3.6.r5b. Abs max $= 0$ at $x = \frac{4}{7}$. Abs min $= -100$ at $x = 2$. 3.6.r5c. Abs max $= -25$ at $x = 5$. Abs min $= -81$ at $x = 3$. 3.6.r5d. Abs max $= 94$ at $x = -2$. Abs min $= -81$ at $x = 3$. 3.6.r5e. Abs max $= 17$ at $x = 1$. Abs min $= 12$ at $x = -2$. 3.6.r5f. Abs max $= \frac{11}{3}$ at $x = 3$. Abs min $= -15$ at $x = 1$. 3.6.r5g. Abs max $= 10$ at $x = 1$. Abs min $= 6$ at $x = 3$. 3.6.r5h. Abs max $= 4$ at $x = -1$ and $x = 2$. Abs min $= 0$ at $x = 1$ and $x = 3$.

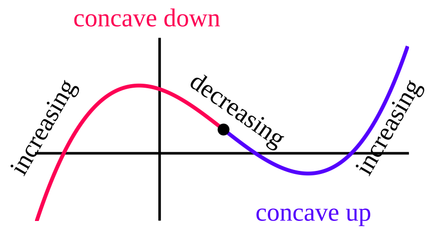
4.1: Concavity and points of inflection

That is, the sign of the first derivative determines a function's **monotonicity** (increasingness and decreasingness). The *second* derivative determines a function's **concavity**:

Definition 4.1.r1. *The graph of f is* **concave up** *when f' is increasing (or $f'' > 0$), and* **concave down** *when f' is decreasing (or $f'' < 0$).*
An **inflection point** *is a point on the graph of f where f changes concavity (and is continuous).*

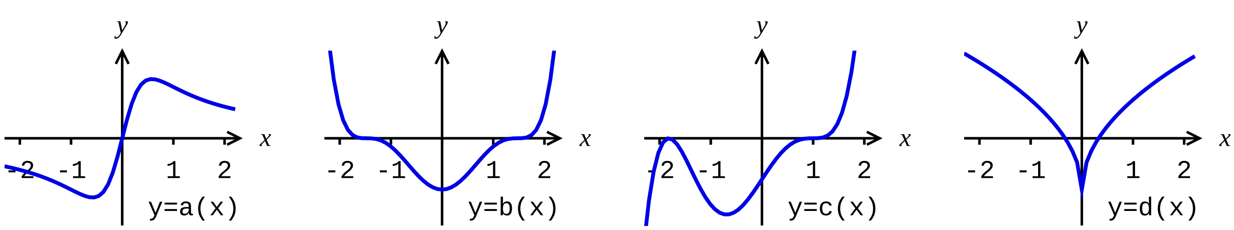
Monotonicity and concavity are independent, in that a function can be any of

increasing and concave up,	increasing and concave down,
decreasing and concave up, or	decreasing and concave down.



Generally, a curve is concave up where its tangent lines lie below the curve, and concave down where its tangent lines lie above the curve. You can see this on an interactive graph of the above function at https://www.desmos.com/calculator/necjpvhc6j

4.1.r2. Estimate the open intervals where the given function is concave up and where it is concave down.



4.1.r3. Find the open intervals where $r(x) = x^4 - 4x^3 - 18x^2$ is concave up and where it is concave down.

Solution: Take the derivative twice to find $r''(x) = 12x^2 - 24x - 36$. To find where $r''(x)$ is positive or negative, first find where it is zero:

$$0 = 12x^2 - 24x - 36 = 12(x^2 - 2x - 3) = 12(x+1)(x-3) \Longrightarrow x = -1 \text{ or } x = 3.$$

Our sign chart for sr'' so far looks like this:

$$\begin{array}{ccc} r''(x): & 0 & 0 \\ \hline x: & -1 & 3 \end{array}$$

To complete the sign chart, determine the sign of $r''(x)$ at sample points chosen to the left of -1, between -1 and 3, and to the right of 3. Tip: it's easiest to use the factorization of r'' found above.

x	-2	0	4
$r''(x)$	$12(-1)(-2)$ > 0	$12(1)(-3)$ < 0	$12(5)(1)$ > 0

Now we know the sign chart completely:

$$\begin{array}{ccc} r''(x): & ++++0----0++++ \\ \hline x: & -1 \qquad 3 \end{array}$$

Therefore, $r(x)$ is concave up on $(-\infty, -1)$ and $(3, \infty)$ and concave down on $(-1, 3)$.

end example 4.1.r3

4.1.r4. Find the open intervals where the given function is concave up and where it is concave down.

a. $1 - 3x + 2x^2$ b. $1 - (3x-2)^3$ c. $x^4 - 6x^3 + 2x - 10$

d. $\dfrac{x^2+9}{x}$ e. $(x+1)^{-1}$ f. $1 - x + 2x^{-2}$

g. $-\frac{1}{12}x^4 - \frac{5}{6}x^3 + 12x^2 + 5x$

Answers

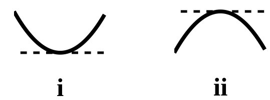
4.1.r2. a. Conc up on $(-1, 0)$ and on $(1, \infty)$. Conc down on $(-\infty, -1)$ and on $(0, 1)$.
b. Conc up on $(-\infty, -1.5)$, on $(-0.5, 0.5)$, and on $(1.5, \infty)$. Conc down on $(-1.5, -0.5)$ and on $(0.5, 1.5)$.
c. Conc up on $(-1.3, 0)$ and on $(1, \infty)$. Conc down on $(-\infty, -1.3)$ and on $(0, 1)$.
d. Conc down on $(-\infty, 0)$. Conc up on $(0, \infty)$. 4.1.r4a. Conc up on $(-\infty, \infty)$.
4.1.r4b. Conc down on $(2/3, \infty)$. Conc up on $(-\infty, 2/3)$. 4.1.r4c. Conc up on $(-\infty, 0)$ and on $(3, \infty)$. Conc down on $(0, 3)$. 4.1.r4d. Conc up on $(0, \infty)$. Conc down on $(-\infty, 0)$.
4.1.r4e. Conc up on $(-1, \infty)$. Conc down on $(-\infty, -1)$ 4.1.r4f. Conc up on $(-\infty, 0)$ and on $(0, \infty)$.
4.1.r4g. Conc up on $(-8, 3)$. Conc down on $(-\infty, -8)$ and on $(3, \infty)$.

4.2: The second derivative test

2nd Derivative Test for Local Extrema 4.2.r1. *Suppose $f'(c) = 0$.*

i. *If $f''(c) > 0$, then $f(c)$ is a local minimum.*

ii. *If $f''(c) < 0$, then $f(c)$ is a local maximum.*



Observe that the Second Derivative Test is inconclusive if $f''(c) = 0$.

In general, the First Derivative Test 3.5.r5 and Second Derivative Test are not useful in determining *absolute* extrema.

4.2.r2. Use the second derivative test to determine where $f(x) = x^4 - 2x^3 - 9x^2$ has its local extrema.

Solution: Local maxima and minima can only occur at critical points. The derivative $f'(x) = 4x^3 - 6x^2 - 18x$ is always defined, so the only critical points are solutions to $f'(x) = 0$. Solve by factoring.

$$f'(x) = 4x^3 - 6x^2 - 18x = 2x(2x + 3)(x - 3)$$

The critial points are $x = 0$, $x = -3/2$, and $x = 0$.

The sign of the second derivative $f''(x) = 12x^2 - 12x - 18 = 6(2x^2 - 2x - 3)$ tells us which of these critical points are minima and which are maxima:

x	$f''(x)$	Conclusion
$-3/2$	27	$f(-3/2)$ is a local minimum.
0	-18	$f(0)$ is a local maximum.
3	54	$f(3)$ is a local minimum.

Here's the graph: https://www.desmos.com/calculator/ic57mct3my

end example 4.2.r2

4.2.r3. Find all critical points and use the second derivative test to classify them as local maxima or local minima.

a. $4 - 5x + x^2$

b. $x^3 - 12x$

c. $4x^3 - 7x^2 + 4x - 1$

d. $2x - x^4$

e. $2 - \frac{1}{x} - 4x$

f. $x^3 + 1$

g. $(x^2 + 1)^{-1}$

h. $(x^3 - 54)/x$

Answers

4.2.r3a. c.p at $x = \frac{5}{2}$. $f''(\frac{5}{2}) = 2 > 0$, so $f(\frac{5}{2})$ is a loc min. 4.2.r3b. c.p. at $x = \pm 2$. $f''(2) = 12 > 0$ and $f''(-2) = -12 < 0$, so $f(2)$ is a loc min and $f(-2)$ a loc max. 4.2.r3c. c.p at $x = \frac{1}{2}, \frac{2}{3}$. $f''(\frac{1}{2}) = -2$, $f''(\frac{2}{3}) = 2$, so $f(\frac{1}{2})$ is a local max, and $f(\frac{2}{3})$ a loc min. 4.2.r3d. Abs max $= 94$ at $x = -2$. Abs min $= -81$ at $x = 3$. 4.2.r3e. c.p. at $x = \pm\frac{1}{2}$. $f''(-\frac{1}{2}) = -16$. $f''(\frac{1}{2}) = 16$. $f(\frac{1}{2})$ is a loc max. $f(-\frac{1}{2})$ is a loc min. 4.2.r3f. c.p. at $x = 0$. $f''(0) = 0$. Second der test is inconclusive. sign chart f' $+++0+++$, so $f(0)$ is not a loc extremum. 4.2.r3g. c.p. at $x = 0$. $f''(0) = 1$. $f(0)$ is a local max. 4.2.r3h. c.p. at $x = 0$. $f''(0) = 0$. Second der test is inconclusive. Sign chart f' $---0+++$ so $f(0)$ a local min.

4.3: Graphing polynomials

In this section, we'll graph functions without calculating more than a few points, using ideas from calculus and precalculus. A good graph shows all of these attributes of the function.

- Domain (the set of all x-values at which $f(x)$ exists)
- Asymptotes
  - $x = c$ is a Vertical Asymptote if $|y| \to \infty$ as $x \to c$.
  - $y = c$ is a Horizontal Asymptote if $y \to c$ as either $x \to \infty$ or $x \to -\infty$.
- Monotonicity
  - $f(x)$ is increasing when $f'(x)$ is positive.
  - $f(x)$ is decreasing when $f'(x)$ is negative.
  - The local extrema of $f(x)$ are where its graph changes monotonicity.
- Concavity
  - The graph of $f(x)$ is concave up when $f''(x)$ is positive .
  - The graph of $f(x)$ is concave down when $f''(x)$ is negative.
  - The infection points of $f(x)$ are where its graph changes concavity.

See page 9 of these notes for the definition of a polynomial.
When graphing polynomials, it's helpful to remember these two facts:

Fact 4.3.r1. *The domain of every polynomial is* $(-\infty, \infty)$. *Its graph has no vertical asymptotes. Unless the polynomial is a constant, its graph has no horizontal asymptotes.*

Fact 2.3.r8. *If* $p(x)$ *is a polynomial and* $\ell t_p(x)$ *is its lead term, then*

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} \ell t_p(x) \qquad \text{and} \qquad \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} \ell t_p(x).$$

4.3.r2. Determine where the polynomial $p(x) = -x^4 + 18x^2$ is increasing, decreasing, concave up, or concave down. Sketch its graph.

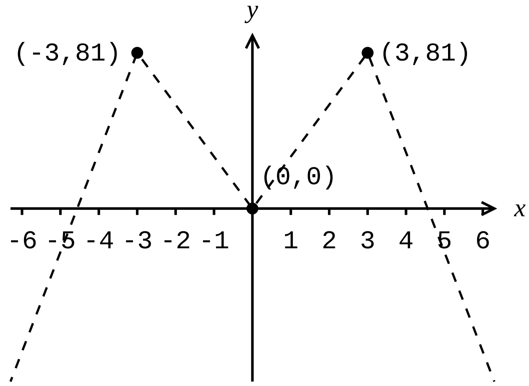
Solution: The derivative $p'(x) = -4x^3 + 36x$ is always defined, so the only critical points are solutions to $p'(x) = 0$. Solve by factoring.

$$p'(x) = -4x(x^2 - 9) = -4x(x - 3)(x + 3)$$

The critial points are $x = -3$, $x = 0$, and $x = 3$. By testing $p'(x)$ at points between these, we produce this sign chart.

$p'(x):$	$+++++0-----0+++++0-----$
$x:$	$-3 \qquad 0 \qquad 3$

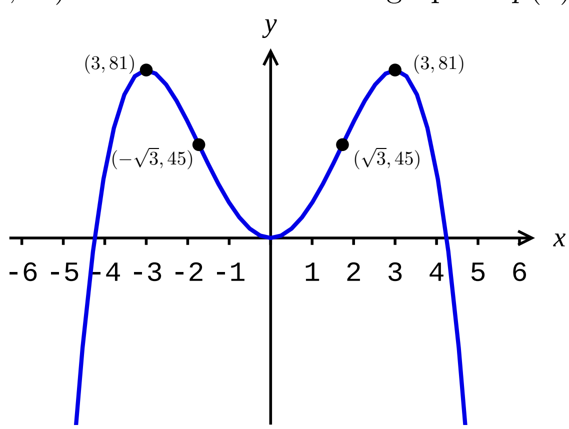
We conclude that $p(x)$ is decreasing on $(-3, 0)$ and on $(3, \infty)$ and increasing on $(-\infty, -3)$ and on $(0, 3)$. The function p has local extrema at $x = -3$, 0, and 3. After calculating $p(3) = 81 = p(-3)$ and $p(0) = 0$, we conclude that graph of p must look roughly like this:



The second derivative $p''(x) = -12x^2 + 36 = -12(x^2 - 3) = -12(x - \sqrt{3})(x + \sqrt{3})$ is zero at $x = \pm\sqrt{3}$. Its sign chart is

$$\begin{array}{r|l} p''(x): & - - - - 0 + + + + + + + 0 - - - \\ \hline x: & \qquad -\sqrt{3} \qquad\qquad\qquad \sqrt{3} \end{array}$$

and so the graph of $p(x)$ is concave down on $(-\infty, -\sqrt{3})$ and on $(\sqrt{3}, \infty)$. The graph is concave up on $(-\sqrt{3}, \sqrt{3})$. Because $p(\pm\sqrt{3}) = 45$, the coordinates of the inflection points are $(-\sqrt{3}, 45)$ and $(\sqrt{3}, 45)$. Now smooth out the graph of $p(x)$ to reflect its concavity:



end example 4.3.r2

4.3.r3. Determine where the given polynomial is increasing, decreasing, concave up, or concave down. Identify its local extrema and inflection points, and sketch its graph. Use Desmos or a graphing calculuator to check your answers.

a. $4 - 6x + x^2$ b. $-\frac{1}{4}x^3 + 3x$ c. $\frac{1}{2}x^4 - 4x^2$

d. $4x^3 - x^4$ e. $(x-2)^2(x+2)^2$

4.3.r4. a. Sketch the graph of function g defined on $(-\infty, \infty)$ satisfying

$$\begin{array}{ll} g(1) = 2 \text{ and } g(3) = 1 & g''(x) > 0 \text{ if } x > 2 \\ g'(x) < 0 \text{ if } x < 1 \text{ or } 3 < x & g''(x) < 0 \text{ if } x < 2 \\ g'(x) > 0 \text{ if } 1 < x < 3 & \end{array}$$

b. Sketch the graph of a function h satisfying

$$\begin{array}{ll} h'(x) \leq 0 \text{ for all } x & h''(x) > 0 \text{ if } x < -1 \\ h(-1) = 1 \text{ and } h'(-1) = 0 & h''(x) < 0 \text{ if } x > -1 \end{array}$$

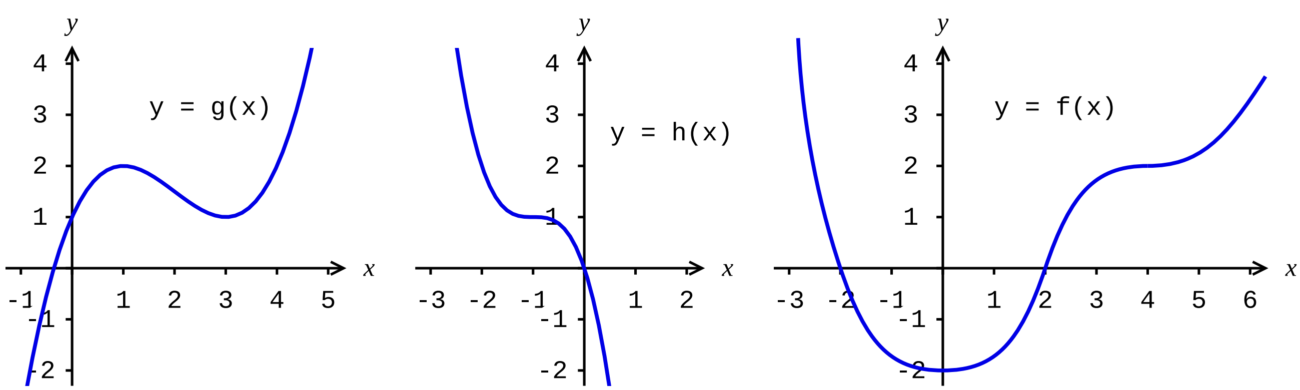
c. Sketch the graph of a function f satisfying

$$\begin{array}{ll} f'(x) < 0 \text{ if } x < 0 & f''(x) < 0 \text{ if } x < -2 \text{ or } 2 < x < 4 \\ f'(x) > 0 \text{ if } 0 < x < 4 \text{ or } 4 < x & f''(x) > 0 \text{ if } -2 < x < 2 \text{ or } 4 < x \\ f'(x) = 0 \text{ at } x = 0, 4 & f''(x) = 0 \text{ at } x = -2, 2, 4 \end{array}$$

Answers

4.3.r3a. dec on $(-\infty, 3)$, inc on $(3, \infty)$. concave up on $(-\infty, \infty)$. $f(3)$ is a loc min. 4.3.r3b. dec on $(-\infty, -2)$ and on $(2, \infty)$, inc on $(-2, 2)$. conc up on $(-\infty, 0)$. conc down on $(0, \infty)$. loc max at $x = 2$. loc min at $x = -2$. inf pt at $x = 0$. 4.3.r3c. dec on $(-\infty, -2$ and on $(0, 2)$, inc on $(-2, 0)$ and on $(2, \infty)$. conc up on $(-\infty, -\sqrt{\frac{4}{3}})$ and on $(\sqrt{\frac{4}{3}}, \infty)$. conc down on $(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}})$. loc max at $x = 0$. loc mins at $x = \pm 2$. inf pts at $x = \pm\sqrt{\frac{4}{3}}$. 4.3.r3d. inc on $(-\infty, 3)$, dec on $(3, \infty)$. conc up on $(0, 2)$, conc down on $(-\infty, 0)$ and on $(2, \infty)$. loc max at $x = 3$. inf pts at $x = 0$ and $x = 2$. 4.3.r3e. inc on $(-2, 0))$ and $(2, \infty)$. dec on $(-\infty, -2)$ and on $(0, 2)$. conc up $(-\infty, -\sqrt{\frac{4}{3}})$ and on $(\sqrt{\frac{4}{3}}, \infty)$. conc down on $(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}})$. loc min at $x = \pm 2$. loc max at $x = 0$. inf pts at $x = \pm\sqrt{\frac{4}{3}}$.

4.3.r4.



4.4: Graphing rational functions

A rational function is the ratio of two polynomials, and it differs form a polynomial is that it may horizontal or vertical asymptotes. See pages 41-43 of these notes.

4.4.r1. Sketch the graph of the function $\rho(x) = x + 4 + \dfrac{4}{x}$.

Solution:
Domain: $\rho(x)$ is defined for all x other than $x = 0$. That is, its domain $= (-\infty, 0) \cup (0, \infty)$.
Asymptotes: Because $\frac{\text{nonzero}}{0}$ indicates an infinite limit, $\lim_{x\to 0^\pm} \rho(x)$ is infinite, so the graph has a vertical asymptote at $x = 0$.
The graph has no horizontal asymptote because the limits

$$\lim_{x\to\pm\infty} x + 4 + \frac{4}{x} = \text{“} \pm \infty + 4 + 0\text{”} = \pm\infty$$

are not finite. In fact, these limits mean

$$\begin{aligned} y \to \infty \quad &\text{as} \quad x \to \infty, \quad \text{and} \\ y \to -\infty \quad &\text{as} \quad x \to -\infty \end{aligned}$$

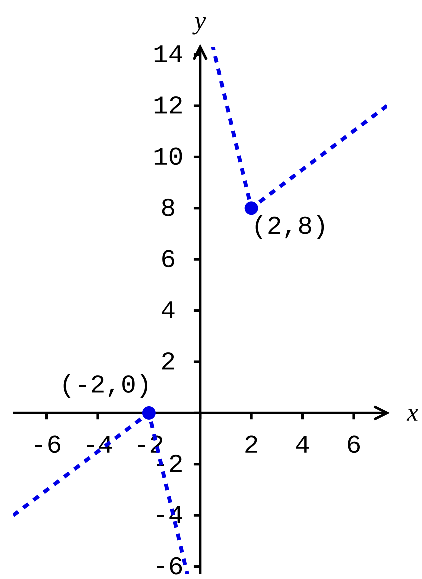
When we draw the graph of ρ on a finite screen, the graph should leave the screen in the upper right and lower left corners.

Monotonicity: $\rho'(x) = (x + 4 + 4x^{-1})' = 1 - 4x^{-2}$. Set $\rho'(x) = 0$ and to find the critical points $x = \pm 2$. Like $\rho(x)$ itself, $rho'(x)$ is undefined at $x = 0$ and may change sign there.

$1 - 4x^{-2}$:	0	DNE	0
x :	-2	0	2

By testing points, we find

$1 - 4x^{-2}$:	$+ + + + 0 - - - -$DNE$- - - - 0 + + + +$
x :	-2 0 2

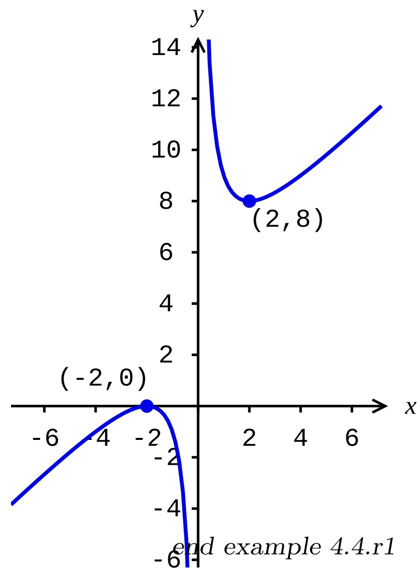


So, ρ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing on the intervals $(-2, 0)$ and $(0, 2)$. ρ has a local maximum at $x = -2$ and a local minimum at $x = 2$. Evaluate ρ at ± 2, plot those points, and rough in the graph based what we know about its intercepts, asymptotes, and monotonicity.

Concavity: $\rho''(x) = 8x^{-3} = \frac{8}{x^3}$. Sign chart:

$$\begin{array}{cc} \frac{8}{x^3}: & ----\text{DNE}++++ \\ \hline x: & 0 \end{array}$$

The graph of ρ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. (Since $\rho(x)$ is undefined at $x = 0$, it has no inflection point at $x = 0$.) Use this concavity information to smooth out your sketch of of the graph. Be careful not to violate the properties of ρ discovered earlier.



end example 4.4.r1

4.4.r2. Give the equations of the horizontal and vertical asympototes of the rational function.

a. $\dfrac{2}{2-x}$ b. $\dfrac{7x-3}{2x+5}$ c. $\dfrac{4x}{x^2-9}$

d. $\dfrac{3x^2+5}{x}$ e. $\dfrac{3x}{x^2+16}$ f. $\dfrac{4x^2-1}{x^2-6x+5}$

4.4.r3. Determine where the given rational function is increasing, decreasing, concave up, or concave down. Give the equations of its vertical and horizontal asymptotes. Identify its local extrema and inflection points. and sketch its graph. Use Desmos or a graphing calculator to check your answers.

a. $\dfrac{-5}{x-6}$ b. $\dfrac{3x-2}{5x+10}$ c. $\dfrac{3x+1}{2x-1}$

d. $x + 16x^{-1}$ e. $2x - 8x^{-1}$ f. $\dfrac{-x^2-25}{x}$

Answers

4.4.r2a. VA is$x = 2$. HA is $y = 0$. 4.4.r2b. VA is $x = -5/2$. HA is $y = 7/2$ 4.4.r2c. VA are $x = 3$ and $x = -3$. HA is $y = 0$. 4.4.r2d. VA is $x = 0$. No HA. 4.4.r2e. no VA. HA is $y = 0$. 4.4.r2f. VA are $x = 5$ and $x = 1$. HA is $y = 4$. 4.4.r3a. VA is $x = 6$, HA is $y = 0$. inc on $(-\infty, 6)$ and on $(6, \infty)$. dec never. concave up on $(-\infty, 6)$ and concave down on $(6, \infty)$. no local extrema or inflection points. 4.4.r3b. VA is $x = -2$, HA is $y = 3/5$. inc on $(-\infty, -2)$ and on $(-2, \infty)$. dec never. concave up on $(-\infty, -2)$ and concave down on $(-2, \infty)$. no local extrema or inflection points. 4.4.r3c. VA is $x = 1/2$, HA is $y = 3/2$. inc never. dec on $(-\infty, 1/2)$ and on $(1/2, \infty)$. dec never. concave down on $(-\infty, 1/2)$ and concave up on $(1/2, \infty)$. no local extrema or inflection points. 4.4.r3d. VA is $x = 0$, no HA. inc on $(-\infty, -4)$ and on $(4, \infty)$. dec on $(-4, 0)$ and on $(0, 4)$. conc up on $(0, \infty)$, conc down on $(-\infty, 0)$ loc max at $x = -4$. local min at $x = 4$. no inf pts. 4.4.r3e. VA is $x = 0$, no HA. dec on $(-\infty, -1/5))$ and $(1/5, \infty)$. inc on $(-1/5, 0)$ and on $(0, 1/5)$. conc down on $(0, \infty)$, conc up on $(-\infty, 0)$ loc min at $x = -1/5$. loc max at $x = 1/5$. no inflection points. 4.4.r3f. VA are $x = 5$ and $x = 1$. HA is $y = 4$.

4.5: Business applications

In this section, we apply methods of calculus to the functions in economics introduced in sections 1.3 and 2.9. Review the definitions in those sections.

4.5.r1. The Jeffersonian is a 40-room luxury hotel. All rooms are occupied when the hotel charges \$300 per night for a room. Only 20 rooms are rented when the hotel charges \$800 per night.Find the demand $D(x)$ for rooms at the Jeffersonian, assuming that it is linear.

*Solution:*Remember that $D(x)$ is the (highest) price at which the Jeffersonian can expect to rent x rooms. For $D(x)$ to be linear means that its graph is a line. Using the given data, we calculate the slope to be

$$\frac{\Delta\text{price}}{\Delta\text{quantity}} = \frac{800-300}{20-40} = -25$$

in point-slope form the line is $D(x) - 300 = -25(x-40)$. Solving for D,

$$D(x) = -25(x-40) + 300,$$

or

$$D(x) = -25x + 1300.$$

4.5.r1, continued. Find the Jeffersonian's revenue function. What number of rooms x will maximize the hotel's revenue, and at what price can the hotel expect to rent that number of rooms?

Solution: Revenue equals price $D(x)$ times quantity x:

$$R(x) = (-25x + 1300)x = -25x^2 + 1300x$$

The graph of R is a concave-down parabola. R is maximized when marginal revenue is zero:

$$R'(x) = -50x + 1300 = 0 \text{ at } x = \frac{1300}{50} = 26.$$

To rent 26 rooms, the hotel should charge

$$D(26) = -25 \cdot 26 + 1300 = \$650$$

dolllar per night.

4.5.r1, continued. Suppose if cost \$100 per night to maintain an occupied room. Find the hotel's profit function $P(x)$. What number of rooms x will maximize the hotel's profit, and at what price can the hotel expect to rent that number of rooms?

*Solution:*The hotel's cost function is $C(x) = 100x$, and profit is revenue minus cost:

$$P(x) = -25x^2 + 1300x - 100x = -25x^2 + 1200x$$

Graph of $P(x)$ is another downward opeing parabola. Maximum profit occurs when marginal profit equals zero:

$$P\text{"}(x) = -50x + 1200 = 0 \text{ at } x = \frac{1200}{50} = 24$$

To rent 26 rooms, the hotel should charge

$$D(24) = -25 \cdot 24 + 1300 = \$700$$

dolllar per night.

end example 4.5.r1

4.5.r2. A discount store sells 84 radios per month at \$40 each. The owners estimate that for each \$2 increase in price, they will sell 3 fewer radios per month. How much should they charge for their radios to maximize their revenue?

4.5.r3. A candy store can sell 180 candy bars at 62 cents each. The store can sell 220 candy bars if the price is 54 cents each. The total cost of producing x candy bars is

$$C(x) = 3000 - 10x + .04x^2$$

cents. Assuming that the demand function is linear, find the number of candy bars that should be produced to maximize profit. What is their selling price at that production level, and what is their average cost per candy bar?

4.5.r4. Given a demand function $D(x) = 20 - x$ and cost function $C(x) = 2x + 21$, where $0 \le x \le 20$.

a. Find the revenue and profit functions.

b. Find the production level x that maximizes profit;

c. Determine average cost at this production level.

d. Determine the profit and selling price at this production level.

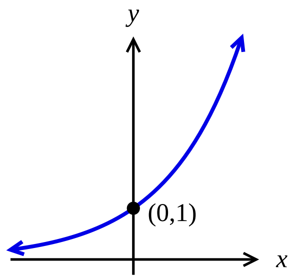
1.3.r1, continued. How many bagels should the bakery produce daily to maximize its profit?

Answers

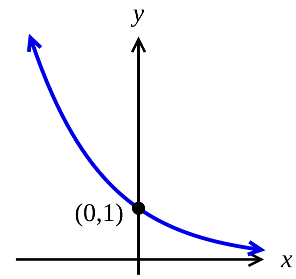
4.5.r2. Owners estimation means that demand is linear: $D(x) = 40 - \frac{2}{3}(x - 84) = 68 - \frac{2}{3}x$. Revenue $= R(x) = xD(x) = 68x - \frac{2}{3}x^2$ is maximized when $R'(x) = 0$, which occurs when $x = 51$ radios. Price at that production level is $D(51) = 34$ dollars. 4.5.r3. $D(x) = 98 - 0.20x$ and $P(x) = 108x - 0.24x^2$. Maximum profit occurs when $P'(x) = 0$, at $x = 225$ candy bars. At that production level, price $= D(225) = 53$ and average cost $= \frac{C(225)}{225} = 3.333$ cents per bar. 4.5.r4a. $R(x) = xD(x) = 20x - x^2$, and $P(x) = R(x) - C(x) = 18x - x^2 - 21$. 4.5.r4b. $x = 9$. 4.5.r4c. Ave Cost $= \frac{C(9)}{9} = 4.333$. 4.5.r4d. Price $= D(9) = 11$. Profit $= P(9) = 60$. 4.5.r4. $x = 2625$ bagels daily.

5.1: Exponential functions

An **exponential function** is one of the form $f(x) = a^x$ for some positive number a. All exponential functions have the same graph, in the sense that the graph of any one can be obtained from that of any other by rescaling in the x-direction and, possibly, reflecting about the y-axis:



$y = a^x$
$a > 1$



$y = a^x$
$0 < a < 1$

The symbol e represents a mathematical constant, roughly equal to 2.718. For reasons we'll go into in section 5.4, e^x is the most commonly occurring exponential function in calculus.

5.1.r1. Make a rough sketch by hand of the graphs of of the following. Label all intercepts and asymptotes. How do the four graphs compare?

a. $y = 2^x$ b. $y = e^x$ c. $y = (\frac{1}{2})^x$ d. e^{-x}

Compound interest

Exponential functions appear in compound interest formulas from finance, using these variables:

$$\begin{aligned} P &= \text{Present value of my investment} \\ F &= \text{Future value of my investment} \\ r &= \text{annual interest rate} \\ n &= \text{number of times that interest is compounded in one year} \\ t &= \text{length of time of the investment, measured in years.} \end{aligned}$$

If you invest P dollars in an account paying r percent annual interest, then after t years, the balance F in your account will be

$$F = P\left(1 + \frac{r}{n}\right)^{nt} \tag{5.1.r2}$$

if interested is **compounded n times a year**, and

$$F = Pe^{rt} \tag{5.1.r3}$$

if the interest is **compounded continuously**. Because

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt}$$

(a fact we prove in Calculus II), continuous compounding can be thought of as the result of compounding interest infinitely many times a year.

5.1.r4. If you deposited \$300 today is an account that pays interest at an annual interest of 6% compounded quarterly, how much would be in your account in 5 years? What if the 6% were compounded monthly? Continuously?

Solution: The question asks us to calculate F in three different scenarios.
Set $P = \$300$, $r = 0.06$, an $t = 5$. In the case of quarterly compounding, $n = 4$, and

$$F = \$300\left(1+\frac{0.06}{4}\right)^{4\cdot 5} = \$300(1.015)^{20} = \$404.06.$$

In the case of monthly compounding, $n = 12$, and

$$F = \$300\left(1+\frac{0.06}{12}\right)^{12\cdot 5} = \$300(1.05)^{60} = \$404.66$$

If interest were compunded continously, then

$$F = \$300e^{0.06*5} = \$300e^{0.3} = \$404.96.$$

end example 5.1.r4

Tip: round your final answer to the nearest penny. To get an accurate final answer, avoid rounding in intermediate steps.

5.1.r4, continued. Find the amount I would have to deposit today in the same account in order to have \$500 in 5 years if interest is compounded

a. 4 times a year
b. continuously

Solution: a. Using $t = 5$, $r = 0.06$, $n = 4$, and $F = \$500$, solve for P in the equation

$$\$500 = P\left(1+\frac{0.06}{4}\right)^{20} = P(1.015)^{20}$$

by dividing both sides by $(1.015)^{20}$:

$$P = \frac{\$500}{(1.015)^{20}} \text{ or } \$500(1.015)^{-20} = \$371.24.$$

b. Set $t = 5$, $r = 0.06$ and $F = \$500$ and solve for P in the equation

$$\$500 = Pe^{0.06*5} = Pe^{0.3}$$

by dividing both sides by $e^{0.3}$:

$$P = \frac{\$500}{e^{0.3}} = \$500e^{-0.3} = \$370.41$$

Since continuous compounding pays more than compounding 4 times per year, I have to deposit more in a. than in b. to reach my \$500 goal.

end example 5.1.r4

5.1.r5. If I deposit \$2,000 today in an account paying 4% interest, what will be the balance in my account after 3 years if interest is compounded

a. semiannually b. quarterly c. monthly d. continuously

5.1.r6. I anticipate needing \$10,000 in 5 years. To achieve this, how much must I deposit today in an account paying 6% if interest is compounded

a. semiannually b. quarterly c. continuouslyi d. continuously

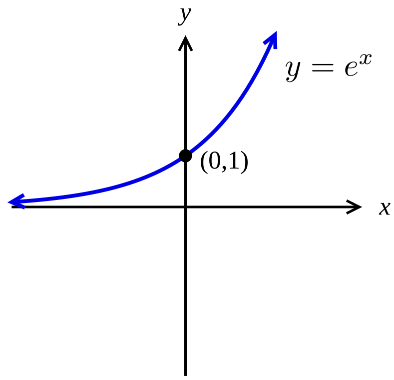
Answers

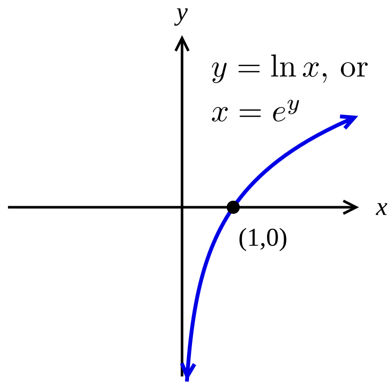
5.1.r1. All four curves have the same y-intercept $(0, 1)$ and no x-intercept. All four lie above the x-axis. $y = 0$ is a horizontal asymptote of 2^x and e^x as $x \to -\infty$ and of e^{-x} and $(\frac{1}{2})^x$ as $x \to +\infty$. $y = e^x$ rises more steeply than $y = 2^x$; $y = e^{-x}$ is the reflection of $y = e^x$ across the y-axis; $y = (\frac{1}{2})^x = 2^{-x}$ is the reflection of $y = 2^x$ across the y-axis. 5.1.r5a. Using $n = 2$, $F = \$2,000(1.02)^6 = \$2,252.32$ 5.1.r5b. $n = 4$, and $F = \$2,000(1.01)^{12}$
$= \$2,253.65$ 5.1.r5c. $F = \$2,000e^{0.12} = \$2254.99$ 5.1.r5d. VA is $x = 0$, no HA. inc on $(-\infty, -4)$ and on $(4, \infty)$. dec on $(-4, 0)$ and on $(0, 4)$. conc up on $(0, \infty)$, conc down on $(-\infty, 0)$ loc max at $x = -4$. local min at $x = 4$. no inf pts. 5.1.r6a. $P = \$10000(1.03)^{-10} = \$7,440.94$
5.1.r6b. $P = \$10000(1.015)^{-20} = \$7,424.70$ 5.1.r6c. $P = \$10000(1.005)^{-60} = \$7,413.72$
5.1.r6d. $P = \$10000e^{-0.3} = \$7,40818$

5.2: Logarithms

The **natural logarithm** is the function defined by the rule

$$\ln b = a \quad \text{means} \quad e^a = b$$





Some important properties of the natural logarithm:

1. $\text{domain} \ln x = (0, \infty) = \text{range}\, e^x$
2. $\text{range} \ln x = (-\infty, \infty) = \text{domain}\, e^x$
3. $x = e^{\ln x}$ for all $x > 0$.
4. $x = \ln(e^x)$ for all real numbers x.
5. $\ln(AB) = \ln A + \ln B$ for all A and $B > 0$.
6. $\ln\left(\dfrac{A}{B}\right) = \ln A - \ln B$ for all A and $B > 0$.
7. $\ln(A^t) = t \ln A$ for all $A > 0$ and for any real t.

Sometimes it is useful to use properties 5-7 to rewrite logarithmic expressions.

5.2.r1. Write

$$\sigma(x) = 3\ln x + 2\ln(x+1) - 4\ln(2x+3)$$

as a single logarithm.

Solution:
First use property 7 to write $\sigma(x) = \ln x^3 + \ln(x+1)^2 - \ln(2x+3)^4$. Now use properties 5 and 6:

$$\sigma(x) = \ln\big(x^3(x+1)^2\big) - \ln(2x+3)^4 = \ln\left(\frac{x^3(x+1)^2}{(2x+3)^4}\right).$$

end example 5.2.r1

5.2.r2. Do the opposite of what you did in example 5.2.r1; break

$$\nu(x) = \log\left(\frac{\sqrt{9x^2+1}}{9x^2+6x+1}\right)$$

into multiple logs with simple arguments.

Solution:
Be careful. $\sqrt{9x^2+1}$ is **not** $3x+1$.
The denominator $9x^2+6x+1 = (3x+1)^2$. Therefore

$$\begin{aligned}\nu(x) &= \log(9x^2+1)^{1/2} - \log(3x+1)^2 \\ &= \frac{1}{2}\log(9x^2+1) - 2\log(3x+1).\end{aligned}$$

The quadratic $9x^2+1$ is irreducible, meaning that it cannot be factored into linear factors using real coefficients, so $\nu(x)$ can't be broken down any further.

end example 5.2.r2

5.2.r3. Combine into a single log.

a. $\ln(2x-1)+\ln(x+1)$
b. $\ln(x+3)-\ln(x^2-4x-21)$
c. $2\log x - \frac{1}{2}\log(x+3)+3\log(x-1)$
d. $\log(x^2-2x)-\log(3x-6)$

5.2.r4. Rewrite in terms of multiple logs with simple arguments.

a. $\ln(12x^3-40x^2+12x)$
b. $\ln\big((3x^2-13x+4)^4\big)$
c. $\ln\left(\frac{x^2-4}{x-2}\right)$

Solving equations with exponentials and logs

We can sometimes use the properties of logs, especially 3 and 4, to solve equations involving exponentials and logs.

5.2.r5. Solve for x.

a. $\ln(3x) - \ln(x+1) = \ln 2$ b. $5^{x-1} = 4$ c. $e^{2x} = -3$

Solution:
a. Combine logs: $\ln\left(\frac{3x}{x+1}\right) = \ln 2$. Since $e^{\ln x} = x$ for all $x > 0$, raise e to both sides to obtain

$$\frac{3x}{x+1} = 2 \quad \Longrightarrow \quad 3x = 2x + 2 \quad \Longrightarrow \quad x = 2$$

b. Take ln of both sides and use property 7 to obtain $(x-1)\ln 5 = \ln 4$, which implies

$$x \ln 5 - \ln 5 = \ln 4 \quad \Longrightarrow \quad x \ln 5 = \ln 4 + \ln 5 \quad \Longrightarrow \quad x = \frac{\ln 4 + \ln 5}{\ln 5}, \text{ or } \frac{\ln 20}{\ln 5}$$

c. No solutions, since e^x is never negative.

end example 5.2.r5

5.2.r6. Solve for x.

a. $e^{x-4} = 2$ b. $\ln(3x-4) = \ln 5$
c. $3^{5x} = 2$ d. $\ln(x+1) + \ln(x-4) = \ln 6$

5.2.r7. If I invest \$100 in an account paying interest at the annual rate of 5%, how long will it take for my investment to grow to \$150 if the interest in my account is compounded

a. 4 times a year? b. continuously?

Solution:
a. Set $P = 100$, $r = 0.05$, $F = 150$, and solve for t in the equation

$$150 = 100(1 + \frac{.05}{4})^{4t} = 100(1.0125)^{4t}$$

Divide both sides by 100 and then take the natural log ln:

$$1.5 = (1.0125)^{4t} \quad \Longrightarrow \quad \ln(1.5) = \ln\big(1.0125^{4t}\big)$$

Now use property 7.:

$$\ln(1.5) = 4t \ln(1.0125) \quad \Longrightarrow \quad t = \frac{\ln(1.5)}{4 \ln(1.0125)} = 8.16.$$

Because 0.16 of a year is $0.16 \cdot 12 = 1.92$ months, it will take a little under 8 years and 2 months for my initial \$100 investment to grow to \$150.

b. Set $P = 100$, $r = 0.05$, $F = 150$. Solve for t in

$$150 = 100e^{.05t} \quad \Longrightarrow \quad 1.5 = e^{0.5t}$$

Take ln of both sides. By property 4.,

$$\ln(1.5) = 0.05t \quad \Longrightarrow \quad t = \frac{\ln(1.5)}{0.05} = 8.11 \text{ years.}$$

end example 5.2.r7

5.2.r8. If I invest \$300 in an account paying 6% interest annually, find the time it will take for my investment to grow to \$600 if the interest in my account is compounded

a. 6 times per year. b. 360 times per year. c. continuously.

Round your answer to 3 places after the decimal.

Answers

5.2.r3a. $\ln(2x-1)(x+1)$, or $\ln(2x^2+x-1)$ 5.2.r3b. $\ln\left(\frac{x+3}{x^2-4x-21}\right) = \ln\left(\frac{1}{x-7}\right)$
5.2.r3c. $\log\left(\frac{x^2(x-1)^3}{\sqrt{x+3}}\right)$ 5.2.r3d. $\log\frac{x}{3}$ 5.2.r4a. $\ln(3x-1)+\ln(x-3)+\ln x+2\ln 2$.
5.2.r4b. $4\ln(3x-1)+4\ln(x-4)$. 5.2.r4c. $\ln(x+2)$. 5.2.r6a. $x = 4+\ln 2$. 5.2.r6b. $x = 3$. 5.2.r6c. $x = \frac{1}{5}(\frac{\ln 2}{\ln 3})$. 5.2.r6d. $x = 5$. (Omit $x = -2$ since both $x+1$ and $x-4$ must be positive.) 5.2.r8a. 11.610 years. 5.2.r8b. 11.553 years 5.2.r8c. 11.552 years.

5.3: Derivative of the natural log

Here's another entry for our catalog of functions and their derivative started on page 52.

$$(\ln x)' = \frac{1}{x}$$

Here's what the Chain Rule looks like when the outer function is the natural log:

$$\big(\ln g(x)\big)' = \frac{1}{g(x)} \cdot g'(x)$$

5.3.r1. Find the derivative of the given function.

a. $\ln 2$ b. $\ln(3x-2)$ c. $x^3\ln(x^2+3)$ d. $2x+\ln(x^3)$ e. $(x+\ln x)^3$

Solution:
a. $\ln 2$ is a constant, so its derivative is 0.
b. By the chain rule, $(\ln(3x-2))' = \frac{1}{3x-2}(3x-2)' = \frac{1}{3x-2}3$, or $\frac{3}{3x-2}$.
c. Product rule

$$\begin{aligned}(x^3\ln(x^2+3))' &= (x^3)'\ln(x^2+3) + x^3(\ln(x^2+3))' \\ &= 3x^2\ln(x^2+3) + x^3\frac{1}{x^2+3}2x \\ &= 3x^2\ln(x^2+3) + \frac{2x^4}{x^2+3}\end{aligned}$$

d. First simply $2x+\ln(x^3) = 2x+3\ln x$. Now differentiate to obtain $2+\frac{3}{x}$.
e. Chain rule:

$$((x+\ln x)^3)' = 3(x+\ln x)^2(x+\ln x)' = 3(x+\ln x)^2\left(1+\frac{1}{x}\right)$$

end example 5.3.r1

5.3.r2. Find the derivative of the given function.

a. $\ln(2x)$ b. $\ln(x^2-3x+2)$ c. $\ln(e^{\sqrt{x}})$

d. $(\ln(2x-3))^4$ e. $\ln\left(\frac{1}{x}\right)$

5.3.r3. Find the absolute max and min of $r(x) = x - \ln(x^2)$ on the interval $[1, 4]$

Solution: Find the critical points of $r(x) = x - 2\ln x$ by solving $r'(x) = 0$:

$$1 - \frac{2}{x} = 0 \quad \Longrightarrow \quad x = 2$$

The absolute max and min can occur only at the endpoints of the interval or critical points in the interval. Compute and compare $r(x)$ at these.

x	1	2	4
$r(x)$	$1 - 2\ln 1$ $= 1$	$2 - 2\ln 2$ $= 0.614$	$4 - 2\ln 4$ $= 1.227$

Therefore, the absolute max must be 1.227 and the absolute min is 0.614.

end example 5.3.r3

Answers

5.3.r2a. $\frac{1}{x}$ 5.3.r2b. $\frac{1}{x-2} + \frac{1}{x-1}$, or $\frac{2x-3}{x^2-3x+2}$.
5.3.r2c. $= (\sqrt{x})' = \frac{1}{2}x^{-1/2}$ 5.3.r2d. $4(\ln(2x-3))^3 \cdot \frac{2}{2x-3}$ 5.3.r2e. $-\frac{1}{x}$

5.4: Derivative of the exponential function

Remarkably, e^x is its own derivative:

$$(e^x)' = e^x$$

Here's what the Chain Rule looks like when the outer function is the exponential function:

$$\left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x)$$

5.4.r1. Find the derivative of the given function.

a. $e^x(2x-1)$ b. $\dfrac{e^x}{x+e^x}$ c. $e^x(x+\ln x)$ d. $e^{x+\ln x}$ e. $(e^x+x)^3$

Solution:

a. Product rule

$$\begin{aligned}(e^x(2x-1))' &= (e^x)'(2x-1)) + e^x(2x-1)' \\ &= e^x(2x-1) + e^x 2 \\ &= e^x(2x-1+2) = e^x(2x+1)\end{aligned}$$

b. Quotient rule

$$\begin{aligned}\left(\frac{e^x}{x+e^x}\right)' &= \frac{(e^x)'(x+e^x) - e^x(x+e^x)'}{(x+e^x)^2} \\ &= \frac{e^x(x+e^x) - e^x(1+e^x)}{(x+e^x)^2}\end{aligned}$$

c. Product rule

$$\begin{aligned}(e^x(x+\ln x))' &= (e^x)'(x+\ln x) + e^x(x+\ln x)' \\ &= e^x(x+\ln x) + e^x(1+x^{-1})\end{aligned}$$

d. Chain rule

$$\begin{aligned}(e^{x+\ln x})' &= e^{x+\ln x}(x+\ln x)' \\ &= e^{x+\ln x}(1+x^{-1})\end{aligned}$$

e. Chain rule:

$$((e^x+x)^3)' = 3(e^x+x)^2(e^x+x)' = 3(e^x+x)^2(e^x+1)$$

end example 5.4.r1

5.4.r2. Find the derivative of the given function.

a. $4x^2e^x$ b. $(e^{x^2} - x^2)^4$ c. $\ln(xe^x)$ d. $e^{(1+x^2)}$

5.4.r3. Find the absolute max and min of $s(x) = (2x-1)e^x$ on the interval $[-1, 2]$

Solution: Find the critical points of $s(x) = (2x-1)e^x$ by solving $s'(x) = 0$:

$$0 = (2x-1)e^x + 2e^x = (2x+1)e^x$$

Since e^x cannot equal zero, the only solution is $x = -\frac{1}{2}$. The absolute max and min can occur only at the endpoints of the interval or critical points in the interval. Compute and compare $s(x)$ at these.

x	-1	-0.5	2
$r(x)$	$-3e^{-1}$ $= -1.104$	$-2e^{-1/2}$ $= -1.902$	$3e^2$ $= 22.167$

Therefore, the absolute max must be 22.167 and the absolute min is -1.902.

end example 5.4.r3

Answers

5.4.r2a. $8xe^x + 4x^2e^x$ 5.4.r2b. $4(e^{x^2} - x^2)^3(2xe^{x^2} - 2x)$ 5.4.r2c. $= (\ln(x) + x)' = x^{-1} + 1$ 5.4.r2d. $2xe^{(1+x^2)}$

6.1: The indefinite integral

An **antiderivative** for $f(x)$ is a function whose derivative is $f(x)$.
If $F(x)$ is an antiderivative of $f(x)$ on an interval, then *every* antiderivative of f on that interval equals $F(x) + C$ for some constant C.
We say that $F(x) + C$ is the **general antiderivative** of f on that interval. The general antiderivative of $f(x)$ is called the **indefinite integral of** $f(x)$, written

$$\int f(x)\,dx$$

6.1.r1. x^2 is an antiderivative of $2x$ because $(x^2)' = 2x$. The indefinite integral of $2x$ is $x^2 + C$. This is written symbolically like this:

$$\int 2x\,dx = x^2 + C$$

I. A catalog of elementary functions and their antiderivatives.

A table of antiderivatives is practically the same as a table of derivatives read backwards:

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	x^{-1}
e^x	e^x

$f(x)$	antiderivative of f
$f(x)$	$\int f(x)\,dx$
x^n	$\frac{1}{n+1}x^{n+1} + C^{[0]}$
x^{-1}	$\ln\lvert x\rvert + C^{[1]}$
e^{kx}	$\frac{1}{k}e^{kx} + C^{[2]}$

More simply, the table on the right says that

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C^{[0]}$$

$$\int \frac{1}{x}\,dx = \ln(|x|) + C^{[1]}$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C^{[2]}$$

Note: [0] is true if and only if $n \neq -1$. Both [1] and [2] can be verified by the chain rule. In [1], $\ln|x|$ is a better antiderivative than $\ln x$ for $\frac{1}{x}$ because $\ln|x|$ is defined on both positive and negative numbers. In [2], k can be any nonzero constant.

II. Combination laws.

1. $\displaystyle\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$
2. $\displaystyle\int \big(f(x)-g(x)\big)\,dx = \int f(x)\,dx - \int g(x)\,dx$
3. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$

In words, 1. says that the antiderivative of a sum is the sum of the antiderivatives. It is important to note that the antiderivative of a product [or quotient] is *not* the product [or quotient] of the antiderivatives.

6.1.r2. x^2 is an antiderivative for $2x$, and e^x is an antiderivative for itself, but x^2e^x is **not** an antiderivative for $2xe^x$, because

$$(x^2e^x)' = 2xe^x + x^2e^x, \text{ not } 2xe^x.$$

$\dfrac{e^x}{x^2}$ is **not** an antiderivative for $\dfrac{e^x}{2x}$, because

$$\left(\frac{e^x}{x^2}\right)' = \frac{x^2e^x - 2xe^x}{x^4} = \frac{e^x(x-2)}{x^3}, \text{ not } \frac{e^x}{2x}.$$

6.1.r3. Find the general antiderivative of the function:

a. $5x^3 - 2\sqrt{x} + 6$ b. $6x^{2/3} + 3x^{5/3} - e^{2x}$ c. $\sqrt{x^3} - 5\sqrt[3]{x^2}$

d. $e^x(1-e^{3x})$ e. $e^{x/3} + \dfrac{2}{x}$ f. $(x-2)(x^{1/2} - x^{-1/2})$

Given the derivative of f, we can only hope to find f up to a constant.

6.1.r4. If $f'(x) = 2x + e^x$, then

$$f(x) = x^2 + e^x + C \qquad (6.1.r5)$$

for some constant C.

To find this constant, we'd also have to know the value of $f(x)$ at one x. For instance, if $f(0) = -2$, then evaluate (6.1.r5) at $x = 0$ to find C:

$$\begin{aligned} f(x) &= x^2 + e^x + C \\ f(0) &= 0^2 + e^0 + C \\ -2 &= 1 + C \end{aligned}$$

so $C = -3$, and $f(x) = x^2 + e^x - 3$.

6.1.r6. Use the given information to find $f(x)$.

a. $f'(x) = 4x^3 - 2e^x + \sec x \tan x; \quad f(0) = 9$

b. $f'(x) = x(3 - 2x^2); \quad f(2) = -3$

c. $f'(x) = \dfrac{5x^2 - 3x + 1}{\sqrt{x}}; \quad f(1) = -6$

6.1.r7. Find the cost function $C(x)$ if marginal cost is $200 + 100e^{-x}$ and fixed cost is \$450.

Answers

6.1.r3a. $\frac{5}{4}x^4 - \frac{4}{3}x^{3/2} + 6x + C$. 6.1.r3b. $\frac{18}{5}x^{5/3} + \frac{9}{8}x^{8/3} - \frac{1}{2}e^{2x} + C$. 6.1.r3c. $\frac{2}{5}x^{5/2} - 3x^{5/3} + C$. 6.1.r3d. Hint: $e^x(1 - e^{3x}) = e^x - e^{4x}$. Antiderivative is $e^x - \frac{1}{4}e^{4x} + C$. 6.1.r3e. $3e^{x/3} + 2\ln|x| + C$. 6.1.r3f. $\frac{2}{5}x^{5/2} - 2x^{3/2} + 4x^{1/2} + C$. 6.1.r6a. $f(x) = x^4 - 2e^x + \sec x + 10$.
6.1.r6b. $f(x) = \frac{3}{2}x^2 - \frac{1}{2}x^4 - 1$ 6.1.r6c. $f(x) = 2x^{5/2} - 2x^{3/2} + 2x^{1/2} - 8$ 6.1.r7. $C(x) = 200x - 100e^{-x} +$ a constant. Use $C(0) = 450$ to find the constant and conclude $C(x) = 200x - 100e^{-x} + 550$.

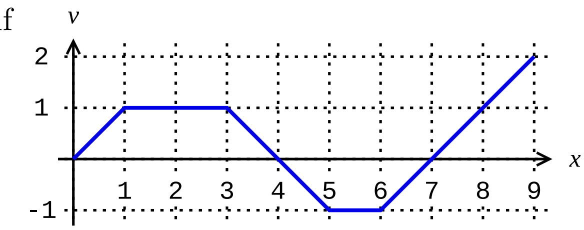
6.4: The definite integral and the Fundamental Theorem of Calculus

The **definite integral from a to b of** $f(x)\,dx$, written

$$\int_a^b f(x)\,dx,$$

is the net signed area between the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.
Area above the x-axis is counted as positive, and area below the x-axis is counted as negative. "Net" refers to the fact that some positive and negative areas may cancel.

6.4.r1. Find the following definite integrals if $v(x)$ is the function pictured here.



a. $\int_0^4 v(x)\,dx$ c. $\int_7^9 v(x)\,dx$

b. $\int_4^7 v(x)\,dx$ d. $\int_0^9 v(x)\,dx$

Solution: Start by finding the areas between $y = v(x)$ and $y = 0$ above/below three intervals on the x-axis. Then treat the area below the x-axis as negative:

x-intervals (sec):	$[0,4]$	$[4,7]$	$[7,9]$
area	3	2	2
signed area	3	-2	2

The first three definite integrals are these signed areas:

a. $\int_0^4 v(x)\,dx = 3$ b. $\int_4^7 v(x)\,dx = -2$ c. $\int_7^9 v(x)\,dx$

The last is the net signed area from $x = 0$ to $x = 9$:

$$d.\ \int_0^9 v(x)\,dx = 3 - 2 + 2 = 3.$$

end example 6.4.r1

The most significant fact in elementary calculus is the relationship between the definite integral and the derivative:

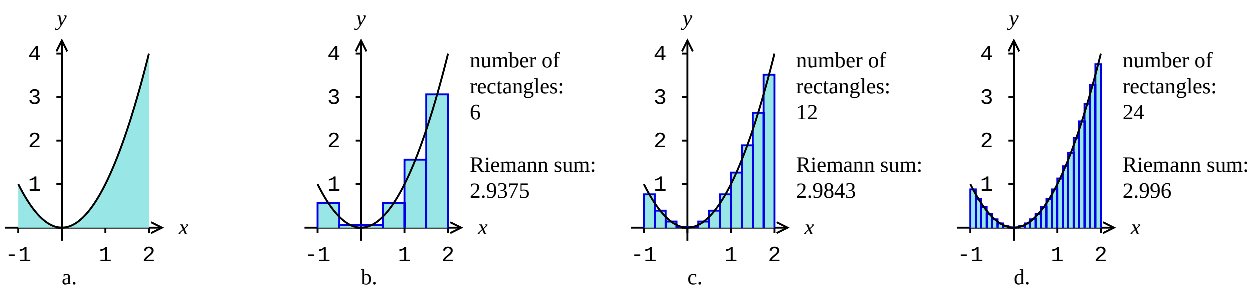
Fundamental Theorem of Calculus (FTC) 6.4.r2. *If $F(x)$ is any antiderivative of $f(x)$ and if f is continuous on $[a, b]$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a) \stackrel{\text{def}}{=} F(x)\Big|_a^b$$

6.4.r3. Since $\frac{1}{3}x^3$ is an antiderivative of x^2, the FTC says that

$$\int_{-1}^{2} (x^2)\,dx = \frac{1}{3}x^3\Big|_{-1}^{2} = \frac{1}{3}2^3 - \frac{1}{3}(-1)^3 = \frac{8}{3} - (-\frac{1}{3}) = 3$$

The definite integral is defined as a limit of "Riemann sums," the sums of areas of rectangles that approximate the area under the curve.



6.4.r3, continued. See the figure above. The definite integral $\int_{-1}^{2}(x^2)\,dx$ equals the shaded area under $y = x^2$ between $x = -1$ and $x = 2$ (a). The FTC tells us that this area is 3. In (b), that area is approximated with 6 rectangles, whose combined area is 2.9375. The number produced this way is called a Riemann sum. (c) and (d) illustrate that when the number of rectangles gets larger, the Riemann sum approaches the definite integral 3.

end example 6.4.r3

6.4.r4. Evaluate the definite integral using the Fundamental Theorem of Calculus.

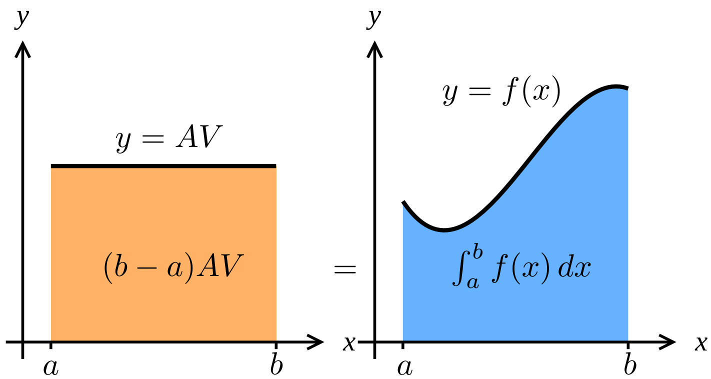
a. $\displaystyle\int_0^2 e^{-2x}\,dx$ b. $\displaystyle\int_{-1}^{1} (3x^2 + x - 2)\,dx$ c. $\displaystyle\int_1^3 \left(e^x + \frac{1}{x}\right)dx$

d. $\displaystyle\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx$ e. $\displaystyle\int_1^4 \left(\frac{1}{\sqrt{x}} + 1\right)(\sqrt{x}+1)\,dx$

Definition 6.4.r5. *The* **average value** *of the function $f(x)$ on the interval $[a, b]$ is*

$$AV = \frac{1}{b-a} \int_a^b f(x)\,dx$$

AV is the altitude of the horizontal line that captures the same area as f over $[a, b]$:



Note that the average value of a given function depends on the interval.

6.4.r6. Calculate the average of x^2 on the given interval.

a. $[-1, 2]$

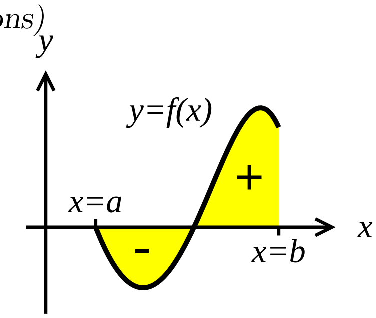
b. $[0, 2]$

6.4.r7. When a turkey cooked to the internal temperature of 66° C is taken out of the oven and set in a 20° C kitchen, its internal temperature x minutes later is $T(x) = 20 + 46e^{-.01x}$. Find the average temperature of the turkey over the first 30 minutes after it's taken from the oven.

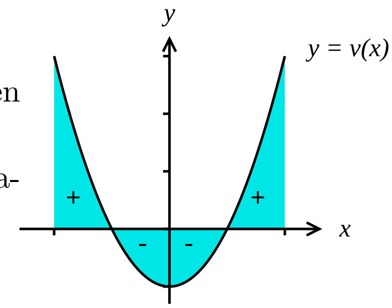
Answers

6.4.r4a. $\frac{1}{2}(1 - e^{-4})$ 6.4.r4b. -2 6.4.r4c. $e^3 - e + \ln 3$ 6.4.r4d. $\frac{134}{5}$ 6.4.r4e. $\frac{10}{3}$ 6.4.r6a. 1 6.4.r6b. 4/3
6.4.r7. $AV = \frac{1}{30}\int_0^{30}(20 + 46e^{-.01x})\,dx = 20 + \frac{460}{3}(1 - e^{-0.3}) = 59.74°$ C.

6.5: The area under a curve (without applications)



The definite integral $\int_a^b f(x)\,dx$, measures the net signed area between the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$. Some positive areas (above the x-axis) may cancel with and negative areas (below the x-axis.



6.5.r1. Let $v(x) = 4x^2 - 1$.

a. Find the *net* area between the graph of $v(x)$ and the x-axis between $x = -1$ and $x = 1$.

The net area is the definite integral $\int_{-1}^{1}(4x^2 - 1)\,dx$. By the Fundamental Theorem of Calculus,

$$\int_{-1}^{1}(4x^2-1)\,dx = (\tfrac{4}{3}x^3 - x)\Big|_{-1}^{1} = (\tfrac{4}{3} - 1) - (-\tfrac{4}{3} + 1) = \tfrac{2}{3}.$$

b. Find the *total* area between the graph of $v(x)$ and the x-axis between $x = -1$ and $x = 1$.

$v(x) = (2x - 1)(2x + 1)$ equals zero at $x = \pm\frac{1}{2}$, so the positive and negative areas in the figure are

$$\int_{-1}^{-1/2}(4x^2-1)\,dx = (\tfrac{4}{3}x^3 - x)\Big|_{-1}^{-1/2} = \tfrac{2}{3}$$

$$\int_{-1/2}^{1/2}(4x^2-1)\,dx = (\tfrac{4}{3}x^3 - x)\Big|_{-1/2}^{1/2} = -\tfrac{2}{3}$$

$$\int_{1/2}^{1}(4x^2-1)\,dx = (\tfrac{4}{3}x^3 - x)\Big|_{1/2}^{1} = \tfrac{2}{3}$$

The net signed area we found in a is

$$\tfrac{2}{3} - \tfrac{2}{3} + \tfrac{2}{3} = \tfrac{2}{3}$$

and the *total* area is

$$\tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3} = 2.$$

end example 6.5.r1

6.5.r2. Find the net signed area between the graph of the given function and the x-axis over the given interval. Then find the total area.

a. $x^3 - 1,\ [0, 2]$ b. $4 - 3x,\ [-2, 1]$ c. $\dfrac{1}{x} - \dfrac{1}{2},\ [1, 4]$

d. $\dfrac{x^2-1}{x},\ [1,2]$ e. $3x^2-2x+1,\ [-1,1]$ f. $(x+1)(x-2),\ [0,3]$

Answers

6.5.r2a. net area $=$ 2. total area $=$ 3.5. 6.5.r2b. net area $=$ 16.5. total area $=$ 16.5. 6.5.r2c. net area $-\frac{3}{2}+\ln 4$. total area $= 3.5$. 6.5.r2d. net area $= -\frac{3}{2}+\ln 4$. total area $= \frac{1}{2}$. 6.5.r2e. net area $= 0$. total area $= \frac{64}{27}$. 6.5.r2f. net area $= -1.5$. total area $= \frac{31}{6}$.