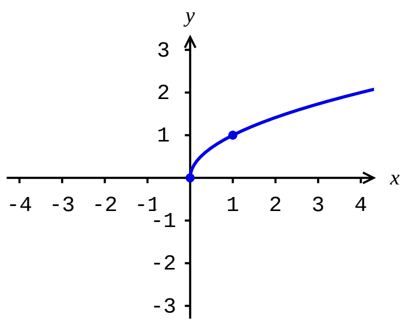
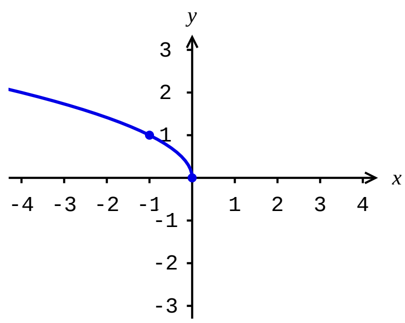
Transformations on the graphs of equations

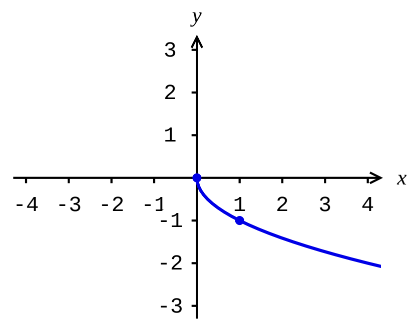
To understand how changes to the equation change that equation's graph, let's look at some examples using $y = \sqrt{x}$. Check for yourself that the coordinates of the points marked with the dot (•) on each graph satisfy the given equation.



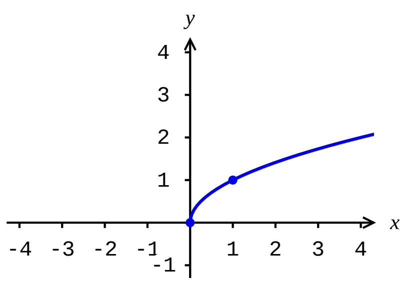
$y = \sqrt{x}$



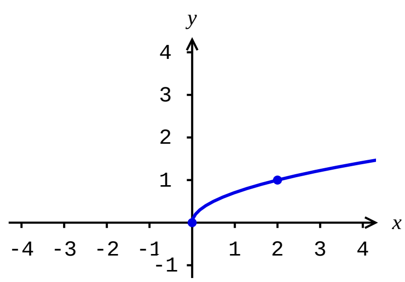
$y = \sqrt{-x}$



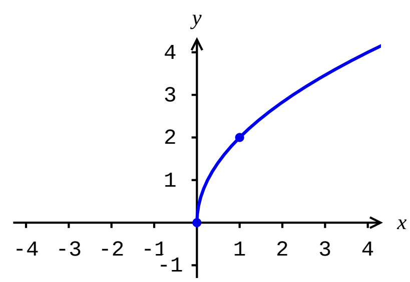
$-y = \sqrt{x}$,
or $y = -\sqrt{x}$



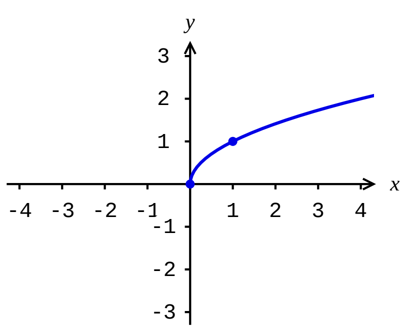
$y = \sqrt{x}$



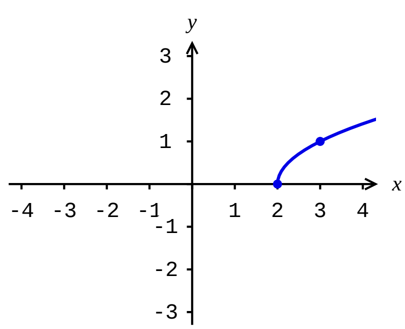
$y = \sqrt{\frac{x}{2}}$



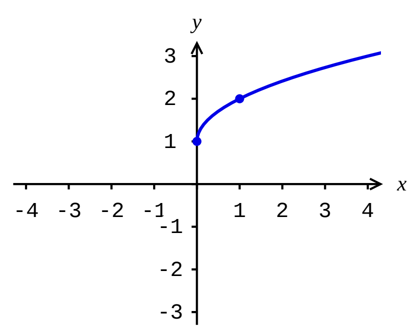
$\frac{y}{2} = \sqrt{x}$,
or $y = 2\sqrt{x}$



$y = \sqrt{x}$



$y = \sqrt{x-2}$



$y - 1 = \sqrt{x}$,
or $y = \sqrt{x} + 1$

(Since the equation $y = \sqrt{x}$ has the form $y = f(x)$, the equations in the third column that result from replacing y by some other expression could also have been obtained instead by replacing the function $f(x)$ by some other function.)

These figures illustrate the following rules. Suppose $E(x, y)$ is an equation in x and y.

Reflections:

1. The graph of $E(-x, y)$ is obtained by reflecting the graph of $E(x, y)$ across $x = 0$.
2. The graph of $E(x, -y)$ is obtained by reflecting the graph of $E(x, y)$ across $y = 0$.

Scaling:

3. The graph of $E(\frac{x}{a}, y)$ is obtained by stretching the graph of $E(x, y)$ horizontally by a factor of a.
4. The graph of $E(x, \frac{y}{b})$ is obtained by stretching the graph of $E(x, y)$ vertically by a factor of b.

Translations:

5. The graph of $E(x - h, y)$ is obtained by shifting the graph of $E(x, y)$ h units right.
6. The graph of $E(x, y - k)$ is obtained by shifting the graph of $E(x, y)$ k units up.

In 5. and 6., "right" and "up" are interpreted according the signs of h and k. For instance, -6 units to the right means 6 units left.

Similar rules apply to the graphs of equations $E(x, y, z)$ in three-space.

When $E(x, y)$ has the form $y = f(x)$,
to replace y with $-y$ is to replace $f(x)$ with $-f(x)$,
to replace y with $\frac{y}{b}$ is to replace $f(x)$ with $bf(x)$, and
to replace y with $y - k$ is to replace $f(x)$ with $f(x) + k$.
For instance, see the three rightmost equations in the previous example.
Therefore, for equations of the form $y = f(x)$, rules 1–6 become

Reflections:

1′. The graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ across $x = 0$.
2′. The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ across $y = 0$.

Scaling:

3′. The graph of $f\left(\frac{x}{a}\right)$ is obtained by stretching the graph of $f(x)$ horizontally by a factor of a.
4′. The graph of $bf(x)$ is obtained by stretching the graph of $f(x)$ vertically by a factor of b.

Translations:

5′. The graph of $f(x - h)$ is obtained by shifting the graph of $f(x)$ h units right.
6′. The graph of $f(x) + k$ is obtained by shifting the graph of $f(x)$ k units up.

To find how the graph of an equation or function is changed by several transformations, it is helpful to picture how to obtain the new equation from the old by a sequence of changes.

Example 1. The function $1 + f(-\frac{1}{2}x - 1)$ can be obtained from $f(x)$ by this sequence:

$$f(x) \xrightarrow{1} f(x-1) \xrightarrow{2} f(\tfrac{1}{2}x - 1) \xrightarrow{3} f(-\tfrac{1}{2}x - 1) \xrightarrow{4} 1 + f(-\tfrac{1}{2}x - 1)$$

1: Replace x by $x - 1$.

2: Replace x by $\frac{1}{2}x$.

3: Replace x by $-x$

4: Add 1 to the resulting function.

So the graph of $1 + f(-\frac{1}{2}x - 1)$ can be obtained from the graph of $f(x)$ by performing the corresponding steps in the same order:

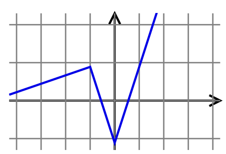
1: Shift the graph 1 unit to the right.

2: Stretch the graph horizontally by the factor 2.

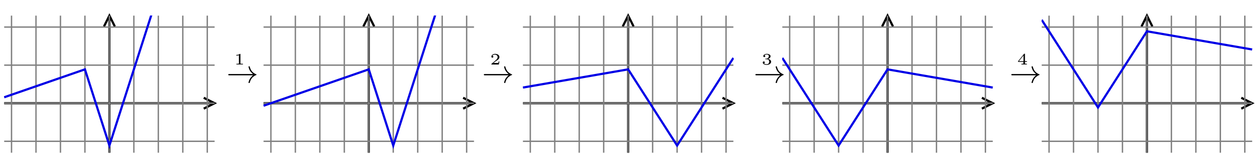
3: Reflect the graph across the line $x = 0$ (the y-axis).

4: Shift the graph one unit up.

For instance, if this was the graph of $y = f(x)$:



then steps 1–4 would look like this:

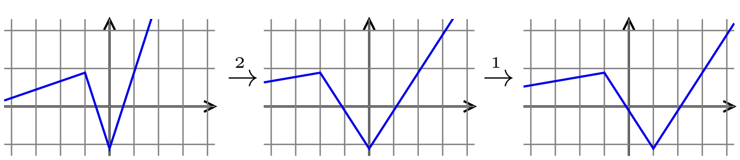


$y = f(x)$ $y = 1 + f(-\frac{1}{2}x - 1)$

In general, order matters. If we had performed step 2 before step 1, the resulting function would have been

$$f(x) \xrightarrow{2} f(\tfrac{1}{2}x) \xrightarrow{1} f(\tfrac{1}{2}(x-1))$$

and its graph would be



1. Review the basic graphs seen on page one of

http://kunklet.people.cofc.edu/MATH120/120review.pdf

Then sketch the graphs of the given functions by hand. Describe in words how each is obtained from the one of the basic graphs. Find any asymptotes and intercepts. Graph on Desmos.com to check your work.

a. $(x-1)^3+1$ b. $\dfrac{-2}{x+1}$ c. $\ln(\frac{1}{3}x)$

d. $\sqrt{1-x}$ e. $6-2|x+1|$ f. $-e^{-x-2}$

Answers

1a. Graph of x^3 is shifted 1 unit right and 1 unit up. Intercept at $(0,0)$. 1b. Graph of $\frac{1}{x}$ is shifted left 1, reflected about y-axis, the stretched vertically by a factor of 2. VA is $x=-1$; HA is $y=0$. Intercept is $(0,-2)$. 1c. Graph of $\ln x$ is stretched horizontally by a factor of 3. VA remains $x=0$. x-intercept is $(3,0)$. 1d. Graph of $\sqrt{x}$ is shifted left 1 unit and then reflected about y-axis. Intercepts are $(1,0)$ and $(0,1)$. 1e. Graph of $|x|$ is shifted left 1, reflected about x-axis, stretched by vertically by a factor of 2, and then shifted up 6 units. Intercepts are $(0,4)$, $(-4,0)$, $(2,0)$. 1f. Graph of e^x is shifted right 2, then reflected about y-axis and about x-axis. Intercept $(0,-e^{-2})$. HA remains $y=0$.